JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 1, February 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.89

A SHORT NOTE ON THE HYERS-ULAM STABILITY IN MULTI-VALUED DYNAMICS

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Abstract. In this paper, we consider the Hyers-Ulam stability on multi-valued dynamics. For a generalized n-dimensional quadratic set-valued functional equation, we prove the Hyers-Ulam stability for the functional equation in multi-valued dynamics.

1. Introduction

The aim of this article is to establish the Hyers-Ulam stability of the generalized quadratic set-valued functional equation. The original stability problem of functional equation concerning group homomorphisms had been first raised by S. M. Ulam [25]. D. H. Hyers [12] gave a first affirmative partial answer to the question of S. M. Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [1] for additive mapping. Th. M. Rassias [22] proved the stability of the linear mapping by being a Cauchy difference of $\varepsilon(\|x\|^p + \|y\|^p)$ for some $\varepsilon \geq 0$ and $0 \leq p < 1$. J. M. Rassias [21] investigated the same problem with $\varepsilon(\|x\|^p \cdot \|y\|^p)$. Thereafter, P. Găvruta [11] provided a generalization of Th. M. Rassias' theorem in which replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$ for the existence of a unique linear mapping. The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the *quadratic functional equation* and every solution of the quadratic functional equation is called a *quadratic function*.

The Hyers-Ulam stability of quadratic functional equation was proved by F. Skof [24] for function $f : E_1 \to E_2$ where E_1 is normed space and

Received January 05, 2018; Accepted January 20, 2018.
2010 Mathematics Subject Classification: 39B82, 47H04, 47H10, 54C60.
Key words and phrases: Hyers-Ulam stability, quadratic set-valued functional equation.
*This research was supported by Physical Metrology for National Strategic Needs funded by Korea Research Institute of Standards and Science (KRISS-2018-GP2018-0005).
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E_2 is a Banach space. P. W. Cholewa [5] extended Skof's theorem by replacing X by an abelian group. Skof's result was generalized by S. Czerwik [10]. He proved the generalized Hyers-Ulam stability of quadratic functional equation in the spirit of Rassias approach. Chu *et al.* [6] extended the quadratic functional equation to the following generalized form

$$_{n-2}\mathrm{C}_{m-2} f(\sum_{j=1}^{n} x_j) + _{n-2}\mathrm{C}_{m-1} \sum_{i=1}^{n} f(x_i) = \sum_{1 \le i_1 < \cdots < i_m \le n} f(x_{i_1} + \cdots + x_{i_m}),$$

where $n \geq 3$ and $2 \leq m \leq n-1$. In [6, 7], they investigated the Hyers-Ulam stability for the generalized quadratic functional equation. Lu and Park [15] defined the additive set-valued functional equations and proved the Hyers-Ulam stability of the set-valued functional equations. Park *et al.* [17] futher investigated stability problems of the Jensen additive, quadratic, cubic and quartic set-valued functional equation. Kenary *et al.* [14] proved the stability for various types of the set-valued functional equation using the fixed point alternative.

In [2], Brzdek investigated some earlier classical results concerning the stability of the additive Cauchy equation. He also disprove a conjecture of Th. M. Rassias and present a new method for proving stability results for functional equations in [3]. In [18, 19], Piszczek obtained some results of stability of functional equation in some classes of multi-valued functions.

Recently, Chu and Yoo [9] investigated the Hyers-Ulam stability of the n-dimensional additive set-valued functional equation. In [8], they also investigated the Hyers-Ulam stability of the n-dimensional cubic set-valued functional equation.

Now we briefly introduce some definitions and notations which are needed to prove main theorems. Let $CB(Y)$ be the set of all closed bounded subsets of Y and $CC(Y)$ the set of all closed convex subsets of Y. Let $CBC(Y)$ be the set of all closed bounded convex subsets of Y. For elements A, B of $CC(Y)$ and $\alpha, \beta \in \mathbb{R}^+$, we denote $A \oplus B := \overline{A + B}$. If A is convex, then we obtain that $(\alpha+\beta)A = \alpha A + \beta A$ for all $\alpha, \beta \in \mathbb{R}^+$. And $_n\mathrm{C}_m$ is defined by $_n\mathrm{C}_m = \frac{n!}{(n-m)!m!}$. Let $f : X \to CBC(Y)$ be a mapping. The *quadratic set-valued functional equation* is defined by

$$f(x+y) \oplus f(x-y) = 2f(x) \oplus 2f(y) \tag{1.1}$$

for all $x, y \in X$. Every solution of the quadratic set-valued functional equation is said to be a *quadratic set-valued mapping*. In the present paper, we define the *generalized n-dimensional quadratic set-valued functional equation* and investigate the Hyers-Ulam-Rassias stability of the functional equation as follows

$$(1.2)\qquad {}_{n-2}\mathrm{C}_{m-2} f\Big(\sum_{j=1}^{n} x_j\Big) \oplus {}_{n-2}\mathrm{C}_{m-1} \sum_{i=1}^{n} f(x_i) = \sum_{1\le i_1<\cdots<i_m\le n} f(x_{i_1}+\cdots+x_{i_m})$$

where $n \geq 3$ and $2 \leq m \leq n-1$. Every solution of the generalized n-dimensional quadratic set-valued functional equation is called a *n-dimensional quadratic set-valued mapping*.

In the next section, to obtain the Hyers-Ulam-Rassias stability of a generalized n-dimensional quadratic functional equation, we use the most popular method induced from the completeness of the phase spaces and another method to gain the stability which is called the *fixed point method*.

Before we deal with the method, we need a terminology. For a set X, we say a function $d : X \times X \to [0, \infty)$ a *generalized metric* on X if d satisfies the following properties:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

2. Stability of the quadratic set-valued functional equation

In this section, we first give basic definitions to prove main theorems and prove the Hyers-Ulam-Rassias stability.

For $A, B \in CB(Y)$, the *Hausdorff distance* $d_H(A, B)$ is defined by

$$d_H(A, B) := inf\{\alpha \geq 0 |\ A \subseteq B + \alpha B_Y, B \subseteq A + \alpha B_Y\},$$

where B_Y is the closed unit ball in Y.

In [4], it was proved that a Hausdorff metric space $(CBC(Y), \oplus, d_H)$ is a complete metric semigroup. Rådström [20] proved that $(CBC(Y), \oplus, d_H)$ is isometrically embedded in a Banach space. To prove main theorems, we need the next remark which states fundamental properties for the Hausdorff distance.

REMARK 2.1. Let $A, A', B, B', C \in CBC(Y)$ and $\alpha > 0$. Then we have that

(1) $d_H(A \oplus A', B \oplus B') \leq d_H(A, B) + d_H(A', B')$;
(2) $d_H(\alpha A, \alpha B) = \alpha d_H(A, B)$;
(3) $d_H(A, B) = d_H(A \oplus C, B \oplus C)$.

Now, we prove the Hyers-Ulam stability of the n-dimensional quadratic set-valued functional equation.

THEOREM 2.2. *Let $n \geq 3$ be an integer and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\sum_{i=0}^{\infty} \frac{1}{4^i} \phi(2^i x_1, \ldots, 2^i x_n) < \infty \tag{2.1}$$

for all $x_1, \ldots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), d_H)$ is an even set-valued mapping with $f(0) = \{0\}$ and

$$d_H\Big({}_{n-2}\mathrm{C}_{m-2} f(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1} \sum_{i=1}^{n} f(x_i), \sum_{1 \leq i_1 < \cdots < i_m \leq n} f(x_{i_1} + \cdots + x_{i_m})\Big) \leq \phi(x_1, \ldots, x_n) \tag{2.2}$$

for all $x_1, \ldots, x_n \in X$. Then for any $m \in \{2, 3, \ldots, n-1\}$, there exists a unique n-dimensional quadratic set-valued mapping $Q : X \to (CBC(Y), d_H)$ such that

$$d_H(f(x), Q(x)) \leq \frac{1}{4 {}_{n-3}C_{m-2}} \sum_{i=0}^{\infty} \frac{1}{4^i} \phi(2^i x, -2^i x, 2^i x, 0, \ldots, 0) \tag{2.3}$$

for all $x \in X$.

Proof. Put $x_1 = x, x_2 = -x, x_3 = x$ and $x_4 = x_5 = \cdots = x_n = 0$ in (2.2). We have

$$\begin{aligned} d_H\Big({}_{n-2}C_{m-2} f(x) &\oplus 3 {}_{n-2}C_{m-1} f(x), 3 {}_{n-3}C_{m-1} f(x) \\ &\oplus {}_{n-3}C_{m-3} f(x) \oplus {}_{n-3}C_{m-2} f(2x)\Big) \\ &\leq \phi(x, -x, x, 0, \ldots, 0) \end{aligned} \tag{2.4}$$

for all $x \in X$. From the condition of remark 2.1, we obtain

$$d_H\big(f(x), \frac{1}{4} f(2x)\big) \leq \frac{1}{4 {}_{n-3}C_{m-2}} \phi(x, -x, x, 0, \ldots, 0) \tag{2.5}$$

for all $x \in X$. Replace x by $2x$ and divide by 4 in (2.5). Then we get

$$d_H\Big(\frac{1}{4}f(2x) \quad , \quad \frac{1}{4^2}f(4x)\Big) \leq \frac{1}{4^2{}_{n-3}C_{m-2}}\phi(2x,-2x,2x,0,\dots,0) \tag{2.6}$$

for all $x \in X$. From (2.5) and (2.6), we obtain

$$d_H\Big(f(x), \frac{1}{4^2}f(4x)\Big) \leq \frac{1}{4_{n-3}C_{m-2}}\phi(x,-x,x,0,\dots,0) + \frac{1}{4^2{}_{n-3}C_{m-2}}\phi(2x,-2x,2x,0,\dots,0) \tag{2.7}$$

for all $x \in X$. Using the induction on i, we get that

$$d_H\Big(f(x), \frac{1}{4^s}f(2^s x)\Big) \leq \frac{1}{4_{n-3}C_{m-2}}\sum_{i=0}^{s-1}\frac{1}{4^i}\phi(2^i x,-2^i x,2^i x,0,\dots,0) \tag{2.8}$$

for any positive integer s and for all $x \in X$.

For all integer r and $l (r > l > 0)$, we have

$$d_H\Big(\frac{1}{4^r}f(2^r x), \frac{1}{4^l}f(2^l x)\Big) \leq \frac{1}{4_{n-3}C_{m-2}}\sum_{k=l}^{r-1}\frac{1}{4^k}\phi(2^k x,-2^k x,2^k x,0,\dots,0) \tag{2.9}$$

for all $x \in X$. Since the right-hand side of the inequality (2.9) tends to zero as k tends to infinity, the sequence $\{\frac{f(2^s x}{4^s}\}$ is a Cauchy sequence in $(CBC(Y), d_H)$. Therefore, we can define a mapping $Q : X \to (CBC(Y), d_H)$ as $Q(x) := \lim_{s\to\infty} \frac{1}{4^s}f(2^s x)$ for all $x \in X$. Now, we show that $Q : X \to (CBC(Y), d_H)$ is a quadratic set-valued mapping. By taking $x_1 = \cdots = x_n = 0$ in (1.2), we have

$$_{n-2}C_{m-2}Q(0) \oplus n_{n-2}C_{m-1}Q(0) = {}_nC_mQ(0).$$

Then we obtain

$$\frac{(m-1)(n-1)!}{m!(n-m-1)!}Q(0) = \{0\}.$$

Since $n \geq 3$, $Q(0) = \{0\}$. By putting $x_1 = x, x_2 = -y, x_3 = y$ and $x_4 = \cdots = x_n = 0$ in (1.2), we have

$$\begin{aligned} {}_{n-2}C_{m-2}Q(x) \oplus {}_{n-2}C_{m-1}Q(x) \oplus 2{}_{n-2}C_{m-1}Q(y) \\ = {}_{n-3}C_{m-2}(Q(x+y) \oplus Q(x-y)) \oplus {}_{n-3}C_{m-3}Q(x) \\ \oplus {}_{n-3}C_{m-1}Q(x) \oplus 2{}_{n-3}C_{m-1}Q(y). \end{aligned}$$

Hence we may have ${}_{n-3}C_{m-2}(Q(x+y) \oplus Q(x-y)) = 2{}_{n-3}C_{m-2}(Q(x) \oplus Q(y))$, that is, Q is a quadratic. Also, a mapping Q satisfies

$$\begin{aligned} {}_{n-2}\mathrm{C}_{m-2}Q(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1}\sum_{i=1}^{n} Q(x_i) \\ = \sum_{1 \leq i_1 < \cdots < i_m \leq n} Q(x_{i_1} + \cdots + x_{i_m}), \end{aligned}$$

where $n \geq 3$ is an integer and $2 \leq m \leq n-1$.

Now, letting $l = 0$ and taking the limit $r \to \infty$ in (2.9), we obtain the inequality (2.3).

To prove the uniqueness of the n-dimensional quadratic set-valued mapping, we assume that $Q' : X \to (CBC(Y), d_H)$ be another n-dimensional quadratic set-valued mapping satisfying (2.3). Then

$$\begin{aligned} (2.10) \quad d_H(Q(x), Q'(x)) &\leq d_H(Q(x), f(x)) \oplus d_H(f(x), Q'(x)) \\ &\leq \frac{2^{1-2r}}{4{}_{n-3}C_{m-2}} \sum_{i=0}^{r-1} \frac{1}{4^i} \phi(2^i x, -2^i x, 2^i x, 0, \ldots, 0) \end{aligned}$$

for all $x \in X$. Taking the limit as $r \to \infty$ in (2.10), we have $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof. □

Remark 2.3. Let $n \geq 3$ be an integer. Consider a change of control function ϕ in the theorem 2.2. Let $\phi : X^n \to [0, \infty)$ be a function such that

$$(2.11) \qquad \sum_{i=0}^{\infty} 4^i \phi(\frac{x_1}{2^i}, \ldots, \frac{x_n}{2^i}) < \infty$$

for all $x_1, \ldots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), d_H)$ is an even set-valued mapping with $f(0) = \{0\}$ and

$$\begin{aligned} (2.12)\ d_H\Big({}_{n-2}\mathrm{C}_{m-2}f(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1}\sum_{i=1}^{n} f(x_i), \\ \sum_{1 \leq i_1 < \cdots < i_m \leq n} f(x_{i_1} + \cdots + x_{i_m})\Big) \leq \phi(x_1, \ldots, x_n) \end{aligned}$$

for all $x_1, \ldots, x_n \in X$. Then for any $m \in \{2, 3, \ldots, n-1\}$, there exists a unique n-dimensional quadratic set-valued mapping $Q : X \to (CBC(Y), d_H)$ such that

$$d_H(f(x), Q(x)) \leq \frac{1}{{}_{n-3}C_{m-2}} \sum_{i=0}^{\infty} 4^i \phi(\frac{1}{2^i}x, -\frac{1}{2^i}x, \frac{1}{2^i}x, 0, \ldots, 0) \tag{2.13}$$

for all $x \in X$.

Corollary 2.4. *Let $n \geq 3$ be an integer, $0 < p < 2$ and $\theta \geq 0$ be real numbers. Suppose that $f : X \to (CBC(Y), d_H)$ is an even mapping satisfying*

$$d_H\Big({}_{n-2}\mathrm{C}_{m-2} f(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1} \sum_{i=1}^{n} f(x_i), \sum_{1 \leq i_1 < \cdots < i_m \leq n} f(x_{i_1} + \cdots + x_{i_m})\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$ and $m \in \{2, 3, \ldots, n-1\}$. Then there exists a unique n-dimensional quadratic set-valued mapping $Q : X \to (CBC(Y), d_H)$ such that

$$d_H(f(x), Q(x)) \leq \frac{1}{2_{n-3}C_{m-2}} \frac{\theta}{4 - 2^p} \|x\|^p$$

for all $x \in X$.

Proof. The result follows theorem 2.2 by setting $\phi(x_1, x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \ldots, x_n \in X$. □

Remark 2.5. By setting $\phi(x_1, x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \ldots, x_n \in X$ in the remark 2.3, we obtain the following statement. Let $n \geq 3$ be an integer, $p > 2$ and $\theta \geq 0$ be real numbers. Suppose that $f : X \to (CBC(Y), d_H)$ is an even mapping satisfying

$$d_H\Big({}_{n-2}\mathrm{C}_{m-2} f(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1} \sum_{i=1}^{n} f(x_i), \sum_{1 \leq i_1 < \cdots < i_m \leq n} f(x_{i_1} + \cdots + x_{i_m})\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$ and $m \in \{2, 3, \ldots, n-1\}$. Then there exists a unique n-dimensional quadratic set-valued mapping $Q : X \to$

$(CBC(Y), d_H)$ such that

$$d_H(f(x), Q(x)) \leq \frac{1}{{}_{2n-3}C_{m-2}} \frac{\theta}{2^p - 4} \|x\|^p$$

for all $x \in X$.

Next we use another method closely related to a fixed point theory to prove the Hyers-Ulam stability of the generalized quadratic set-valued functional equation. We first introduce a useful theorem to prove our results. In [16], the following lemma is due to Margolis and Diaz.

LEMMA 2.6. *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \ \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

Generally, the fixed point method is so popular technique to prove the Hyers-Ulam stability. In the set-valued version, we also use this useful method to prove the Hyers-Ulam stability.

THEOREM 2.7. *Let $2 \leq m \leq n-1$ be an integer. Suppose that an even mapping $f : X \longrightarrow (CBC(Y), d_H)$ with $f(0) = \{0\}$ satisfies the inequality*

(2.14)
$$d_H\Big({}_{n-2}\mathrm{C}_{m-2} f(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1} \sum_{i=1}^{n} f(x_i), \sum_{1 \leq i_1 < \cdots < i_m \leq n} f(x_{i_1} + \cdots + x_{i_m})\Big) \leq \phi(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in X$ and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0, \infty)$ satisfies

$$\phi(2x, -2x, 2x, 0, \ldots, 0) \leq 4L\phi(x, -x, x, 0, \ldots, 0) \tag{2.15}$$

for all $x \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $Q : X \to (CBC(Y), d_H)$ such that

$$d_H\big(f(x), Q(x)\big) \le \frac{1}{4(1-L)}\phi(x, -x, x, 0, \dots, 0) \tag{2.16}$$

for all $x \in X$.

Proof. Set $x_1 = x, x_2 = -x, x_3 = x$ and $x_4 = \cdots = x_n = 0$ in (2.14). Since f is even and the range of f is convex, we have that

$$d_H\big(f(x), \frac{1}{4}f(2x)\big) \le \frac{1}{4_{n-3}C_{m-2}}\phi(x, -x, x, 0, \dots, 0) \tag{2.17}$$

for all $x \in X$.

Let $S := \{g \mid g : X \to CBC(Y),\ g(0) = \{0\}\}$. We define a generalized metric on S defined by

$$d(g_1, g_2) := \inf\Big\{\mu \in (0, \infty) \mid d_H(g_1(x), g_2(x)) \le \mu\phi(x, -x, x, 0, \dots, 0), x \in X\Big\},$$

where, as usual, $inf\emptyset := \infty$.

Now, we define the mapping $J : (S, d) \to (S, d)$ given by $Jg(x) = \frac{1}{4}g(2x)$ for all $x \in X$. For $g_1, g_2 \in S$, let $d(g_1, g_2) < \mu$. Then

$$d_H\big(g_1(x), g_2(x)\big) \le \mu\phi(x, -x, x, 0, \dots, 0)$$

for all $x \in X$. By (2.15), we have

$$\begin{aligned} d_H\big(Jg_1(x), Jg_2(x)\big) &= \frac{1}{4}d_H(g_1(2x), g_2(2x)) \\ &\le \frac{1}{4}\mu\phi(2x, -2x, 2x, 0, \dots, 0) \\ &\le L\mu\phi(x, -x, x, 0, \dots, 0) \end{aligned}$$

for all $x \in X$.

Therefore, we have that $d(Jg_1, Jg_2) \le Ld(g_1, g_2)$ for all $g_1, g_2 \in S$. Hence J is a strictly contractive mapping with the Lipschitz constant L. From (2.17), we can obtain that $d(f, Jf) \le \frac{1}{4}$. By theorem 2.6, there exists a unique fixed point $Q : X \to (CBC(Y), d_H)$ of J such that $\{J^r f\} \to 0$ as $r \to \infty$. Then we have

$$Q(x) = \lim_{r\to\infty} \frac{1}{4^r} f(2^r x) \tag{2.18}$$

for all $x \in X$. Also, from the fixed point alternative, we get $d(f, Q) \le \frac{1}{1-L}d(Jf, f) \le \frac{1}{4(1-L)}$, which implies the inequality (2.16) holds.

From (2.14) and (2.18), it follows that

$$d_H\Big({}_{n-2}\mathrm{C}_{m-2}Q(\textstyle\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1}\displaystyle\sum_{i=1}^{n} Q(x_i), \sum_{1\le i_1<\cdots<i_m\le n} Q(x_{i_1}+\cdots+x_{i_m})\Big) \le \lim_{r\to\infty}\frac{1}{4^r}\phi(2^r x_1,\ldots,2^r x_n) = 0$$

for all $x_1,\ldots,x_n \in X$.

Therefore, Q is a unique n-dimensional quadratic set-valued mapping as desired. □

REMARK 2.8. Let $2 \le m \le n-1$ be an integer. Suppose that an even mapping $f : X \longrightarrow (CBC(Y), d_H)$ with $f(0) = \{0\}$ satisfies the inequality

$$d_H\Big({}_{n-2}\mathrm{C}_{m-2}f(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1}\sum_{i=1}^{n} f(x_i), \sum_{1\le i_1<\cdots<i_m\le n} f(x_{i_1}+\cdots+x_{i_m})\Big) \le \phi(x_1,\ldots,x_n)$$

for all $x_1,\ldots,x_n \in X$ and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0,\infty)$ satisfies

$$\phi(\frac{x}{2},-\frac{x}{2},\frac{x}{2},0,\ldots,0) \le \frac{L}{4}\phi(x,-x,x,0,\ldots,0)$$

for all $x \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $Q : X \to (CBC(Y), d_H)$ such that

$$d_H\big(f(x),Q(x)\big) \le \frac{L}{4-4L}\phi(x,-x,x,0,\ldots,0)$$

for all $x \in X$.

COROLLARY 2.9. *Let $0 < p < 2$ and $\theta \ge 0$ be real number. Suppose that $f : X \longrightarrow (CBC(Y), d_H)$ is an even mapping satisfying*

$$d_H\Big({}_{n-2}\mathrm{C}_{m-2}f(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1}\sum_{i=1}^{n} f(x_i), \sum_{1\le i_1<\cdots<i_m\le n} f(x_{i_1}+\cdots+x_{i_m})\Big) \le \theta\sum_{i=1}^{n}\|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$. *Then there exists a unique n-dimensional quadratic set-valued mapping* $Q : X \to (CBC(Y), d_H)$ *such that*

$$d_H\big(f(x), Q(x)\big) \leq \frac{3\theta}{2^2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from theorem 2.7 by setting $\phi(x_1, x_2, \ldots, x_n) = \theta\sum_{i=1}^{n}\|x_i\|^p$ for every $x_1, \ldots, x_n \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired results. □

Remark 2.10. In remark 2.8, we set $\phi(x_1, x_2, \ldots, x_n) = \theta\sum_{i=1}^{n}\|x_i\|^p$ for every $x_1, \ldots, x_n \in X$. Then we obtain the following statement. Let $p > 2$ and $\theta \geq 0$ be real number. Suppose that $f : X \longrightarrow (CBC(Y), d_H)$ is an even mapping satisfying

$$d_H\Big({}_{n-2}\mathrm{C}_{m-2}f(\sum_{j=1}^{n} x_j) \oplus {}_{n-2}\mathrm{C}_{m-1}\sum_{i=1}^{n} f(x_i), \sum_{1\leq i_1<\cdots<i_m\leq n} f(x_{i_1} + \cdots + x_{i_m})\Big) \leq \theta\sum_{i=1}^{n}\|x_i\|^p$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $Q : X \to (CBC(Y), d_H)$ such that

$$d_H\big(f(x), Q(x)\big) \leq \frac{\theta}{2^p - 2^2}\|x\|^p$$

for all $x \in X$.

Competing interesting : The authors declare that they have no competing interests.

Authors contributions : All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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