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CHOOSING REGULARIZATION PARAMETER BY GLOBAL *L*-CURVE CRITERION

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Abstract. As an efficient way to determine the regularization parameter in the discrete ill-posed problems with multiple right-hand sides, we suggest global L-curve criterion as an extension of L-curve technique for image restoration problems with single right-hand side.

1. Introduction

Let $H \in R^{N\times N}$ be an imaging system and $B \in R^{N\times s}$ $(N \gg s)$ be a collection of the column stacking of each small blocks obtained by partitioning the blurred and noisy images. The image deblurring in reconstruction problem is stated by the following large-scale inverse problems with multiple right-hand sides

$$HX = B.$$

The multiple right-hand sides B is contaminated by an error $\mathcal{E}$ which may stem from either discretization or measurement inaccuracies. Let $\overline{B}$ denote the unavailable error-free images of the object B,

$$B = \overline{B} + \mathcal{E}.$$

Typically H is a large full rank matrix, having singular values which accumulate at the origin and gradually decay to zero, so it is difficult to determine its numerical rank. This ill-posed nature of the problem may give rise to significant errors in computing approximations of the true image solution.

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Because of severely ill-conditioning of the matrix H and the presence of errors, the Tikhonov regularization method is applied to approximate the true image solution in deblurring problems. In general, space-invariant imaging systems with multiple right-hand sides are often modeled as equivalent minimization problem shown in the following:

$$\min_X(\|HX - B\|_F^2 + \lambda^2 \|X\|_F^2) \tag{1.1}$$

([10, 11]). The effectiveness of F-norm based Tikhonov regularization strongly depends on the choice of the regularization parameter λ balancing the tradeoff between the smoothness of the solution and the fidelity to the observed data where an appropriate value of this parameter λ is not known a priori.

By applying the preconditioned global conjugate gradient linear least squares (PGl-CGLS) method to image deblurring problems (1.1), it evidently shows that there are great improvements in execution times[11]. The global generalized cross validation(GCV) technique is suggested to determine the better regularization parameter λ in PGl-CGLS method[12]. To obtain more accurate approximation of the true images in deblurring problems, [1] adapts the weighted global GCV function to determine the Tikhonov parameter λ.

For 2-norm based image deblurring problems, i.e. the number of column stacks $s = 1$ in (1.1), the determination of the regularization parameter λ is from Morozov's discrepancy principle, L-curve criterion, generalized cross validation, and new variants of these methods are suggested in [3, 5, 6, 14].

The L-curve, the plot of the norm of the regularized solution versus the corresponding residual norm for each set of regularization parameter values, was introduced by Lawson and popularized by Hansen [3, 4, 9]. There is an analysis for the shape of L-curve and theoretical justification in choosing the regularization parameter [2, 13]. The underlying idea is that a good method for choosing the regularization parameter for discrete ill-posed problem must incorporate information about the solution size along with the residual size. It becomes indeed quite natural to seek a fair balance while keeping both of these values small. The L-curve has a distinct L-shaped corner located exactly where the solution x_λ changes in nature from being dominated by regularization errors to being dominated by the errors in the right-hand side. Hence the corner of L-curve corresponds to a good balance between minimization of the sizes, and the corresponding regularization parameter λ is an ideal solution.

This paper will suggest global L-curve criterion as selection of regularization parameter in image deblurring problems (1.1) based on the classical L-curved method.

The sequel is organized as follows: the brief description of the L-curve criterion for the regularization problem with single right-hand side is summarized in Section 2. Section 3 describes the global L-curve method implemented for the image deblurring problems with multiple right-hand sides. Results of numerical experiments including the analytical comments are given in Section 4 and 5.

2. Review of the *L*-curve

The L-curve was introduced by Lawson and popularized by Hansen [9, 3]. The L-curve method was developed for the selection of regularization parameters in the solution of discrete systems obtained from ill-posed problems with single right-hand side,

$$\min_x (\|Hx - b\|_2^2 + \lambda^2 \|x\|_2^2), \tag{2.1}$$

which can be obtained by considering $s = 1$ in (1.1). The unique solution of (2.1) is $x_\lambda = (H^T H + \lambda^2 I)^{-1} H^T b$. Using the singular value decomposition of $H = \sum_{i=1}^N u_i \sigma_i v_i^T$, the regularized solution x_λ is given by

$$x_\lambda = \sum_{i=1}^N \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \frac{u_i^T b}{\sigma_i} v_i.$$

Setting $\alpha_i = u_i^T b$, the norm of the solution x_λ and the residual norm for x_λ are given by

$$\xi = \|x_\lambda\|_2^2 = \sum_{i=1}^N \frac{\sigma_i^2 \alpha_i^2}{(\sigma_i^2 + \lambda^2)^2}$$

and

$$\rho = \|b - Hx_\lambda\|_2^2 = \sum_{i=1}^N \frac{\lambda^4 \alpha_i^2}{(\sigma_i^2 + \lambda^2)^2}.$$

As functions of regularization parameter λ, $\|x_\lambda\|_2$ and $\|b - Hx_\lambda\|_2$ are decreasing and increasing respectively since $\frac{d\xi}{d\lambda} = -4\lambda \sum_{i=1}^N \frac{\sigma_i^2 \alpha_i^2}{(\sigma_i^2+\lambda^2)^3}$ and $\frac{d\rho}{d\lambda} = 4\lambda^3 \sum_{i=1}^N \frac{\sigma_i^2 \alpha_i^2}{(\sigma_i^2+\lambda^2)^3}$. Thus there exists a function $\lambda(\rho)$ inverse to ρ. Since $\frac{d\xi}{d\rho} = -\frac{1}{\lambda^2}$, the solution norm $\|x_\lambda\|_2$ is monotonically decreasing function of the residual norm $\|b - Hx_\lambda\|_2$. Also, $\xi(\rho)$ is strictly convex from the fact that $\frac{d^2\xi}{d\rho^2} = \frac{2}{\lambda^3}\frac{d\lambda}{d\rho} > 0$.

If λ becomes too large then $\|b - Hx_\lambda\|_2$ also becomes large while $\|x_\lambda\|_2$ becomes small. Thus the problem we are solving has only a little connection with the original equation. Hence the parameter λ controls how much the weight is given to minimization of $\|x_\lambda\|_2$ in relation to the minimization of the residual $\|b - Hx_\lambda\|_2$. To find a good balance between these two terms, via suitable value of λ is expected so that the regularized solution be a good approximation of the true solution.

A convenient way to display information about the regularized solution x_λ to (2.1), as a function of the regularization parameter λ, is to plot the norm $\|x_\lambda\|_2$ of the solution versus the residual norm $\|b - Hx_\lambda\|_2$ for each set of regularization parameter values. *L*-curve is the continuous curve consisting of all points $(\|b - Hx_\lambda\|_2, \|x_\lambda\|_2)$ for $\lambda \in [0, \infty)$.

A good regularization parameter λ is one that corresponds to a regularization solution near the "corner" of the *L*-curve since in this region there is a good compromise between achieving a small residual norm $\|b - Hx_\lambda\|_2$ while keeping the solution norm $\|x\|_2$ reasonably small.

Intuitively, the best regularization parameter should lie on the corner of the *L*-curve, since for values higher than the best regularization parameter, the residuals increase without reducing the norm of the solution much. On the other hand, for values smaller than the best regularization parameter, the norm of the solution increases rapidly without much decrease in residual. In practice, only a few points on the *L*-curve are computed and the corner is located by estimating the point of maximum curvature[4].

3. Global *L*-curve criterion for regularization parameters

An additional analysis of the *L*-curve criterion illustrated in the last section for *F*-norm based on linear least squares problems (1.1) with multiple right-hand sides is stated.

The regularization solution of (1.1) is expressed as $X_\lambda = (H^TH + \lambda^2 I)^{-1}H^TB$. To take account of the singular value decomposition of $H = U\Sigma V^T$, the regularization solution can be transformed into

$$X_\lambda = \begin{pmatrix} \sum_{i=1}^N \frac{\sigma_i}{\sigma_i^2+\lambda^2}[U_i^TB_1]V_i & \sum_{i=1}^N \frac{\sigma_i}{\sigma_i^2+\lambda^2}[U_i^TB_2]V_i & \cdots & \sum_{i=1}^N \frac{\sigma_i}{\sigma_i^2+\lambda^2}[U_i^TB_s]V_i \end{pmatrix}$$

where $U = \begin{bmatrix} U_1 & U_2 & \dots & U_N \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & V_2 & \dots & V_N \end{bmatrix}$. The residual matrix also comes to be

$$HX_\lambda - B = U(\Sigma(\Sigma^2 + \lambda^2 I)^{-1}\Sigma - I)U^T B.$$

Taking F-norm, we then acquire

$$\widetilde{\xi} = \|X_\lambda\|_F^2 = \sum_{j=1}^{s}\sum_{i=1}^{N}\left(\frac{\sigma_i^2}{\sigma_i^2+\lambda^2}\frac{[U_i^T B_j]}{\sigma_i}\right)^2 \tag{3.1}$$

and

$$\widetilde{\rho} = \|HX_\lambda - B\|_F^2 = \sum_{j=1}^{s}\sum_{i=1}^{N}\left(\frac{\lambda^2}{\sigma_i^2+\lambda^2}[U_i^T B_j]\right)^2. \tag{3.2}$$

THEOREM 3.1. *For the Tikhonov regularized solution X_λ in (1.1), $\|X_\lambda\|_F$ is a monotonically decreasing convex function of $\|HX_\lambda - B\|_F$, and*

$$0 \le \|HX_\lambda - B\|_F \le \|B\|_F\,, \quad 0 \le \|X_\lambda\|_F \le \|X_{LS}\|_F\,.$$

Proof. From the formula (3.1)

$$\widetilde{\xi}\,' = \frac{d\widetilde{\xi}}{d\lambda} = \sum_{j=1}^{s}\sum_{i=1}^{N} -\frac{4}{\lambda}\left(\frac{\sigma_i^2}{\sigma_i^2+\lambda^2}\right)^2 \frac{\lambda^2}{\sigma_i^2+\lambda^2}\left(\frac{U_i^T B_j}{\sigma_i}\right)^2. \tag{3.3}$$

Since $\widetilde{\xi}\,' < 0$ for all λ, $\widetilde{\xi}$ is decreasing. From the formula (3.2)

$$\widetilde{\rho}\,' = \frac{d\widetilde{\rho}}{d\lambda} = \sum_{j=1}^{s}\sum_{i=1}^{N} \frac{4}{\lambda}\left(\frac{\lambda^2}{\sigma_i^2+\lambda^2}\right)^2 \frac{\sigma_i^2}{\sigma_i^2+\lambda^2}\left(U_i^T B_j\right)^2. \tag{3.4}$$

Since $\widetilde{\rho}\,' > 0$ for all λ, $\widetilde{\rho}$ is increasing. Thus there exists a function $\lambda(\widetilde{\rho})$ inverse to $\widetilde{\rho}$. From (3.3) and (3.4) we obtain

$$\frac{d\widetilde{\xi}}{d\widetilde{\rho}} = -\frac{1}{\lambda^2} < 0.$$

Therefore the solution norm $\|X_\lambda\|$ is monotonically decreasing function of the residual norm $\|B - HX_\lambda\|$. Since $\lambda(\widetilde{\rho})$ is increasing,

$$\frac{d^2\widetilde{\xi}}{d\widetilde{\rho}^2} = \frac{2}{\lambda^3}\frac{d\lambda}{d\widetilde{\rho}} > 0.$$

This means $\widetilde{\xi}(\widetilde{\rho})$ is strictly convex. □

Moreover, a point $(\widehat{\rho}, \widehat{\xi})$ on the curve $(\|HX_\lambda - B\|_F, \|X_\lambda\|_F)$ is a solution to the following two inequality-constrained least squares problems:

$$\widehat{\rho} = \min_X \|HX - B\|_F \text{ subject to } \|X\|_F \leq \widehat{\xi},\ 0 \leq \widehat{\xi} \leq \|X_{LS}\|_F, \tag{3.5}$$

$$\widehat{\xi} = \min_X \|X\|_F \quad \text{subject to} \quad \|HX - B\|_F \leq \widehat{\rho},\ 0 \leq \widehat{\rho} \leq \|B\|_F. \tag{3.6}$$

Let $\overline{X}_\lambda = (H^T H + \lambda^2 I)^{-1} H^T \overline{B}$ be the regularized version of the exact solution $\overline{X} = H^\dagger \overline{B}$ for (1.1), and $X_\lambda^{\mathcal{E}} = (H^T H + \lambda^2 I)^{-1} H^T \mathcal{E}$ be the solution matrix which is achieved by implementing Tikhonov regularization to the pure noise component $\mathcal{E}$ of the multiple right-hand sides. Then the Tikhonov solution for (1.1) can be rewritten as $X_\lambda = \overline{X}_\lambda + X_\lambda^{\mathcal{E}}$.

In order to analyze the L-curve with respect to the exact data $\overline{B}$, assume that the exact SVD coefficients $|U_i^T \overline{B}_j|$ go gradually to zero at faster speed than the σ_i for $j = 1, 2, ..., s$. This assumption confirms that the least squares solution $\overline{X}$ to the noise-free problem, thus, does not have a large norm since the exact solution coefficients $\left\|V_i^T \overline{X}\right\|_2^2 = \sum_{j=1}^s (U_i^T \overline{B}_j / \sigma_i)^2$ which also go to zero. It also gives a physically meaningful solution to the underlying inverse problem. The solution can be estimated by a regularized solution on the condition that an appropriate regularization parameter can be found. See the details in [2].

Assume that the regularization parameter λ is located somewhere between σ_1 and σ_N, such that we have both some small filter factors, $\frac{\sigma_i^2}{\sigma_i^2 + \lambda^2}$, and some filter factors close to 1. Let k indicate the number of filter factors close to 1. Then, it is clear that k and λ are related by the expression $\lambda \simeq \sigma_k$. Since the coefficients $\left\|V_i^T \overline{X}\right\|_2$ go to zero such that the last $N - k$ terms influence the sum at very minimal,

$$\left\|\overline{X}_\lambda\right\|_F^2 \simeq \sum_{i=1}^k \left\|V_i^T \overline{X}\right\|_2^2 \simeq \sum_{i=1}^N \left\|V_i^T \overline{X}\right\|_2^2 = \left\|\overline{X}\right\|_F^2. \tag{3.7}$$

The above expression holds if λ is not too large. As $\lambda \to \infty$ and $k \to 0$, $\overline{X}_\lambda \to O$ and thus $\left\|\overline{X}_\lambda\right\|_F \to 0$. As $\lambda \to 0$, $\overline{X}_\lambda \to \overline{X}$ and thus $\left\|\overline{X}_\lambda\right\|_F \to \left\|\overline{X}\right\|_F$. Then, the residual for $\overline{X}_\lambda$:

$$\left\|H\overline{X}_\lambda - \overline{B}\right\|_F^2 \simeq \sum_{j=1}^s \sum_{i=k+1}^N (U_i^T \overline{B}_j)^2,$$

shows that the residual norm increases at a stable speed 0 to $\|\overline{B}\|_F$ as λ increases steadily. Therefore, the L-curve for the noise-free problem is an overall flat curve at $\|\overline{X}_\lambda\|_F \simeq \|\overline{X}\|_F$, with exceptions to large values of the residual norm $\|H\overline{X}_\lambda - \overline{B}\|_F$ where the curve gets closer to the abscissa axis.

For considering an L-curve corresponding to multiple right-hand side constructed with pure noise $\mathcal{E}$, assume that the noise $\mathcal{E}$ is that the covariance matrix for j-th column $\mathcal{E}_j$ is a scalar times the identity matrix. Then the expected values of the SVD coefficients $u_i^T\mathcal{E}_j$ become independent of i,

$$E((U_i^T\mathcal{E}_j)^2) = \epsilon^2, i = 1, ..., N,\ j = 1, ...s.$$

On the other hand, by considering the norm of $X_\lambda^{\mathcal{E}}$ we get

$$\begin{aligned}\|X_\lambda^{\mathcal{E}}\|_F^2 &\simeq \sum_{j=1}^{s}\sum_{i=1}^{N}\left(\frac{\sigma_i\epsilon}{\sigma_i^2+\lambda^2}\right)^2 \\ &\simeq \sum_{j=1}^{s}\left(\sum_{i=1}^{k}(\frac{\epsilon}{\sigma_i})^2 + \sum_{i=k+1}^{N}(\frac{\sigma_i\epsilon}{\lambda^2})^2\right) \\ &= \sum_{j=1}^{s}\epsilon^2\left(\sum_{i=1}^{k}\sigma_i^{-2} + \lambda^{-4}\sum_{i=k+1}^{N}\sigma_i^2\right).\end{aligned}$$

Since $\sum_{i=1}^{k}\sigma_i^{-2}$ is dominated by $\sigma_k^{-2} \simeq \lambda^{-2}$ while $\sum_{i=k+1}^{N}\sigma_i^2$ is dominated by $\sigma_{k+1}^2 \simeq \lambda^2$, the following approximate expression can be obtained:

$$\|X_\lambda^{\mathcal{E}}\|_F \simeq c_\lambda\epsilon\sqrt{s}/\lambda,$$

where c_λ is a quantity that changes at a slow speed with respect to λ. Thus, the norm of $X_\lambda^{\mathcal{E}}$ increases monotonically from 0 as long as λ decreases until it reaches the value $\|H^\dagger\mathcal{E}\|_F \simeq \epsilon\|H^\dagger\|_F$ for $\lambda = 0$.

From the norm of the corresponding residual

$$\|HX_\lambda^{\mathcal{E}} - \mathcal{E}\|_F^2 \simeq \sum_{j=1}^{s}\sum_{i=k+1}^{N}\epsilon^2 = (N-k)s\epsilon^2,$$

assure that $\|HX_\lambda^{\mathcal{E}} - \mathcal{E}\|_F \simeq \sqrt{(N-k)s}\epsilon$ is slowly changing function of λ located in the range from 0 to $\|\mathcal{E}\| = \sqrt{Ns}\epsilon$. The L-curve regards to $\mathcal{E}$ is an overall very steep curve located slightly left of $\|HX_\lambda^{\mathcal{E}} - \mathcal{E}\|_F \simeq \|\mathcal{E}\|_F$, with exception of some small values of λ approaching the ordinate axis.

Lastly, lets consider the L-curve with respect to the noisy right-hand side $B = \overline{B} + \mathcal{E}$. As shown, it is either the noise-free components $U_i^T\overline{B}$

or the pure-noise components $U_i^T\mathcal{E}$ that dominate depending on the value of λ. Hence, the resulting L-curve consists of one *leg* from the unperturbed L-curve and one *leg* from the pure-noise L-curve. Note that for small values of λ it is the pure-noise L-curve that controls since X_λ is controlled by $X_\lambda^{\mathcal{E}}$, and for large values of λ where X_λ is dominated by $\overline{X}_\lambda$ it is the unperturbed L-curve that dominates. Somewhere in between the two curves, there is a range of λ-values that correspond to a transition between the two dominating L-curves.

The L-curve in the logarithmic scale highlights the difference between the L-curves for an exact right-hand side $\overline{B}$ and for pure noise $\mathcal{E}$, and it further highlights the two opposing parts of the L-curve for multiple noisy right-hand sides $B = \overline{B} + \mathcal{E}$.

If the function $\widetilde{\rho}$ and $\widetilde{\xi}$ are defined by some computable formula, and if the L-curve is twice continuously differentiable, then it is straightforward to compute the curvature $\widetilde{\kappa}_\lambda$ of the L-curve by means of the formula

$$\widetilde{\kappa}_\lambda = \frac{\widetilde{\rho}\,'\widetilde{\xi}\,'' - \widetilde{\rho}\,''\widetilde{\xi}\,'}{\left[(\widetilde{\rho}\,')^2 + (\widetilde{\xi}\,')^2\right]^{3/2}} \tag{3.8}$$

since $\widetilde{\rho}\,' = -\lambda^2\widetilde{\xi}\,'$, $\widetilde{\rho}\,'' = \frac{d\widetilde{\rho}\,'}{d\lambda} = \frac{d}{d\lambda}(-\lambda^2\widetilde{\xi}\,') = -2\lambda\widetilde{\xi}\,' - \lambda^2\widetilde{\xi}\,''$. The curvature $\widetilde{\kappa}_\lambda$ in (3.8) becomes

$$\widetilde{\kappa}_\lambda = \frac{2\lambda(\widetilde{\xi}\,')^2}{\left[(\widetilde{\rho}\,')^2 + (\widetilde{\xi}\,')^2\right]^{3/2}}. \tag{3.9}$$

Therefore the curvature $\widetilde{\kappa}_\lambda > 0$ for all λ indicating the curve $(\widetilde{\rho}, \widetilde{\xi})$ is convex. Any one dimensional optimization routine can be used in locating the value of λ with respect to maximum curvature.

4. Numerical experiments

We investigated numerical results to illustrate the effectiveness of the regularization parameters chosen from the maximization of the curvature of the global L-curve. The image deblurring problems are solved by algorithm combined with the global L-curve technique in preconditioned Gl-CGLS method which is summarized below:

Algorithm 1. Preconditioned Gl-CGLS with the global L-curve criterion

1. Determine the maximizer λ_{gL} for the constrained maximization problem:

$$\max_{\lambda} \quad \widehat{\kappa}_\lambda \qquad \text{subject to} \quad \sigma_N \leq \lambda \leq \sigma_1$$

2. Solve the preconditioned problem $\Omega^{-T}(H^TH+\lambda_{gL}^2 I)X = \Omega^{-T}H^TB$:

   i. $R_0 = \begin{pmatrix} B \\ O \end{pmatrix} - \begin{pmatrix} H \\ \lambda_{gL}I \end{pmatrix} X_0$, $P_0 = S_0 = \Omega^{-T}\begin{pmatrix} H \\ \lambda_{gL}I \end{pmatrix}^T R_0$, $\gamma_0 = (S_0, S_0)_F$.

   ii. For $k = 0, 1, 2, ...$ until $\|R_k\|_F \,/\, \|R_0\|_F \le tol$

   (i) $T_k = \Omega^{-1}P_k$ $Q_k = \begin{pmatrix} H \\ \lambda_{gL}I \end{pmatrix} T_k$, $\alpha_k = \gamma_k/(Q_k, Q_k)_F$,

   (ii) $X_{k+1} = X_k + \alpha_k T_k$, $R_{k+1} = R_k - \alpha_k Q_k$,

   (iii) $S_{k+1} = \Omega^{-T}\begin{pmatrix} H \\ \lambda_{gL}I \end{pmatrix}^T R_{k+1}$, $\gamma_{k+1} = (S_{k+1}, S_{k+1})_F$,

   (iv) $\beta_k = \gamma_{k+1}/\gamma_k$, $P_{k+1} = S_{k+1} + \beta_k P_k$.

   Enddo

In order to implement the above algorithm a few programs needed for our test are developed by improving certain routines from Hansen's Regularization Tools[7]. We ran the algorithm with Matlab for four test images named as *x*, *grain*, *text1* and *text2* and calculated the relative F-norm difference between the regularized solution and the exact solutions. The exact, noise-free solutions are known in these examples.

The first image has a size of 32-by-32 and others has a size of 256-by-256. These are degraded by *Gaussian* blur and Gaussian noises are added. Gaussian blurring parameter is set to 2 for *x*, *text1* and 3 for *grain*, *text2*. As the 256-by-256 images are divided into the collection of smaller blocks using 32×32 sub-blocks, we have image deblurring problems with 64 multiple right-hand sides. Each blocks of test images include 0.5%, 0.9%, and 1% in noise level. All images are applied with reflective boundary condition and so the preconditioner Ω in Algorithm 1 is set to $\Omega = \mathcal{C}^T(|\Lambda_H|^2 + \lambda_{gL}^2 I)^{1/2}\mathcal{C}$ where $H = \mathcal{C}^T\Lambda_H\mathcal{C}$ and $\mathcal{C}$ is two dimensional discrete cosine transformation matrix.

In order to get the local maximizer λ_{gL} of curvature function κ_λ of global L-curve, we computed the minimum negative of the curvature κ_λ in (3.9) by using the matlab function *fminbnd* to find a minimum of single-variable function on fixed interval.

For the purpose of analysis of comparison results, we computed the relative accuracy of approximated solution X_k, $\|X^* - X_k\|_F/\|X^*\|_F$, one of the measures for the approximation of the true image X^*, and show them along the global L-curve criterion and PSNR in Table 1. The approximated solutions were obtained by preconditioned Gl-CGLS method with regularization parameters chosen from the global L-curve criterion. The corner of the global L-curve for the Tikhonov were determined using a new function **lcorner_g**, a modification of Hansen's

Image	Noise level(%)	λ_{gL}	Relative accuracy / PSNR
x	0.5	0.1969	4.71e-001 /59.54
grain		0.1758	2.70e-001 /58.83
text1		0.1087	1.67e-001 /52.51
text2		0.2089	7.33e-001 /51.08
x	0.9	0.1970	4.63e-001 /59.70
grain		0.1759	2.56e-001 /59.30
text1		0.1089	1.65e-001 /52.61
text2		0.2089	7.08e-001 /51.38
x	1	0.1970	4.57e-001 /59.81
grain		0.1759	2.41e-001 /59.83
text1		0.1090	1.64e-001 /52.67
text2		0.2089	6.87e-001 /51.65

TABLE 1

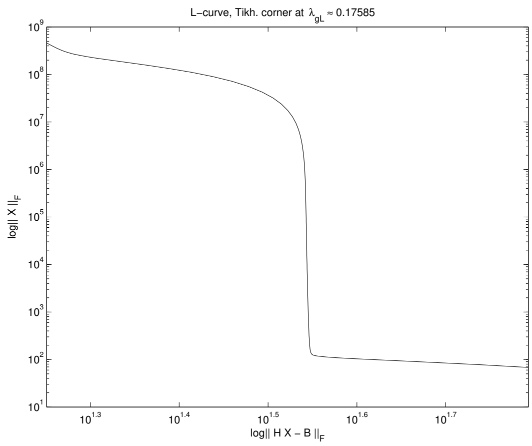

FIGURE 1. Global L-curve for *grain* image: A log-log plot of the solution norm $\|X_\lambda\|_F$ versus the residual norm $\|R_\lambda\|_F$ with λ as the parameter.

lcorner function. The stopping rule of the preconditioned Gl-CGLS iteration is the current residual R_k that satisfies the criteria $\frac{\|R_k\|_F}{\|R_0\|_F} \leq tol$, where tol is set to 10^{-2}.

For *grain* image with 0.5% noise level, regularization parameter λ_{gL} from global L-curve criterion can be chosen 0.1758 which lies near the global L-curve's corner. The Tikhonov global L-curve is shown in Figure 1. With $\lambda_{gL} \approx 0.1758$, the approximated solution got the relative

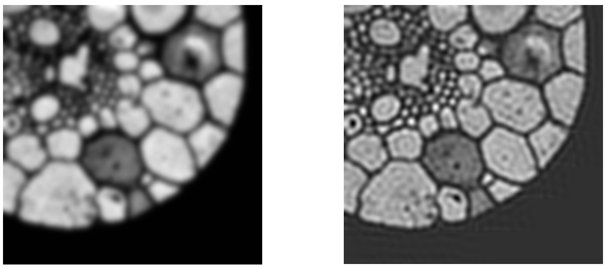

FIGURE 2. (a) Blurred and noisy image (b) Restored image using global L-curve criterion.

accuracy of 0.270413 and PSNR of 58.83. Corresponding blurred and noisy image and the restored image are shown in Figure 2.

5. Conclusions

To obtain more accurate approximation of the true images in deblurring problems, we have proposed an efficient way to determine the regularization parameter by applying the global L-curve criterion to the inverse problem with multiple right-hand sides. We also have analyzed the experimental results for four test images.

In the future, our next study is to compare our study with both global L-curve criterion and global GCV method to obtain the improved true images in deblurring problems.

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