JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **30**, No. 3, August 2017
http://dx.doi.org/10.14403/jcms.2017.30.3.323

NOTE ON SPECTRUM OF LINEAR DIFFERENTIAL OPERATORS WITH PERIODIC COEFFICIENTS

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Abstract. In this paper, by rigorous calculations, we consider $L^2(\mathbb{R})$-spectrum of linear differential operators with periodic coefficients. These operators are usually seen in linearization of nonlinear partial differential equations about spatially periodic traveling wave solutions. Here, by using the solution operator obtained from Floquet theory, we prove rigorously that $L^2(\mathbb{R})$-spectrum of the linear operator is determined by the eigenvalues of Floquet matrix.

1. Introduction

In this note, we characterize the spectrum of the following n-th order linear differential operator with periodic coefficients

(1.1)
$$Lu := (D\partial_x^n + A_{n-1}(x)\partial_x^{n-1} + A_{n-2}(x)\partial_x^{n-2} \cdots + A_1(x)\partial_x + A_0(x))u,$$

where $u : \mathbb{R} \longrightarrow \mathbb{C}^m$, $m \geq 1$, $D \in \mathbb{R}^{m\times m}$ is a constant nonsingular matrix, and the coefficients matrices $A_j(x) \in \mathbb{R}^{m\times m}$ are continuous π-periodic, that is, $A_j(x+\pi) = A_j(x)$ for all $x \in \mathbb{R}$. Here, we consider the operator L on $L^2(\mathbb{R})$ with densely defined domain $H^n(\mathbb{R})$. This differential operator typically arises from the linearization of nonlinear partial differential equations about spatially periodic solutions. As an example, the Swift-Hohenberg equation

$$u_t = -(1+\partial_x^2)^2 u + \varepsilon^2 u - u^3, \quad (t \geq 0, \quad x \in \mathbb{R}) \tag{1.2}$$

has a family of stationary periodic solutions $\bar{u}(x)$ for sufficiently small $\varepsilon^2 > 0$ (see [8] for detailed form of solutions). If we linearize (1.2)

Received March 27, 2017; Accepted April 28, 2017.

2010 Mathematics Subject Classification: 35P05.

Key words and phrases: spectrum, linear operator with periodic coefficients, Floquet theory.

*Supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (No. 2016R1C1B1009978).

about $\bar{u}$ we obtain a 4-th order linear differential operator with periodic coefficients $L := -(1+\partial_x^2)^2 + \varepsilon^2 - 3\bar{u}^2(x)$.

Characterization of the spectrum of (1.1) plays an important role in study of stability of periodic solutions. Nonlinear stability of spectrally stable periodic traveling waves have been widely studied in [1, 2, 3, 4, 5, 6] for the system of reaction diffusion equations and of conservation laws by using the fact that $L^2(\mathbb{R})$-spectrum of the linear operator (1.1) is entirely essential spectrum. The purpose of this note is to provide a rigorous proof of the fact for general $n, m \geq 1$.

The spectrum of exponentially asymptotic linear operators of the form (1.1) without periodicity of coefficients have been characterized rigorously in [7] for $m = 1$. Here we modify their prove for the linear operator with periodic coefficients.

We begin by writing definitions of resolvent set and spectrum of L, denoted by $\rho(L)$ and $\sigma(L)$.

Definition 1.1. We say that $\lambda \in \mathbb{C}$ lies in the resolvent set of L, that is, $\lambda \in \rho(L)$ if $L - \lambda I$ has a bounded inverse. In other words, there is a constant $C > 0$ such that for any $f \in L^2(\mathbb{R})$, there exists $u \in H^n(\mathbb{R})$ which satisfies $(L - \lambda I)u = f$ and $\|u\|_{H^n(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}$. We define spectrum as $\sigma(L) = \mathbb{C} \setminus \rho(L)$.

1.1. The first order ODE system

In order to characterize $L^2(\mathbb{R})$-spectrum of L, we need to consider an eigenvalue problem

$$\lambda u = Lu, \tag{1.3}$$

which can be written as the following first order ODE system

$$U' = \mathbb{A}(x,\lambda)U, \tag{1.4}$$

where

(1.5)

$$U = \begin{pmatrix} u & u' & \cdots & u^{(n-2)} & Du^{(n-1)} \end{pmatrix}^t,$$

$$\mathbb{A}(x,\lambda) = \begin{pmatrix} 0_m & I_m & 0_m & \cdots & 0_m \\ 0_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_m & 0_m & 0_m & \cdots & I_m \\ \lambda I_m - A_0(x) & -A_1(x) & -A_2(x) & \cdots & -A_{n-1}(x) \end{pmatrix}.$$

Here, the superscript "t" means "transpose" of a matrix. By recalling $u \in \mathbb{C}^m$, we notice that $U \in \mathbb{C}^{mn}$, $0_m \in \mathbb{R}^{m\times m}$, and $\mathbb{A}(x,\lambda) \in \mathbb{C}^{mn\times mn}$.

Since $A_j(x)$ is π-periodic, $\mathbb{A}(x+\pi,\lambda) = \mathbb{A}(x,\lambda)$. We now state the Floquet theory for (1.4) which is a key theory for characterizing the spectrum of linear differential operator with periodic coefficients. We denote the fundamental matrix solution of (1.4) by $\Psi(x,\lambda)$. See [7] for the proof.

THEOREM 1.2 (Floquet's Theorem, [7]). *If $\mathbb{A}(x,\lambda) \in \mathbb{C}^{mn\times mn}$ is continuous π-periodic, then the fundamental matrix solution $\Psi(x,\lambda)$ for the system* (1.4) *has the form*

$$\Psi(x,\lambda) = P(x,\lambda)e^{R(\lambda)x}, \tag{1.6}$$

where $R(\lambda) \in \mathbb{C}^{mn\times mn}$ is a constant matrix and $P(x,\lambda) \in \mathbb{C}^{mn\times mn}$ is continuous π-periodic in x.

Without loss of generality, one can assume $\Psi(x,\lambda)$ is the principle fundamental matrix solution, that is, $\Psi(0,\lambda) = P(0,\lambda) = I_{mn}$. Throughout this paper, we denote by $\mathbb{E}^s$, $\mathbb{E}^u$ and $\mathbb{E}^c$ the stable eigenspace, unstable eigenspace and center eigenspace of the constant matrix $R(\lambda)$, respectively. Since $\mathbb{C}^{mn} = \mathbb{E}^s \oplus \mathbb{E}^u \oplus \mathbb{E}^c$, each vector $V \in \mathbb{C}^{mn}$ can be decomposed into $V = V_s + V_u + V_c$ for some $V_{s,u,c} \in \mathbb{E}^{s,u,c}$. We now set

$$\mathbb{P}^{s,u,c} : \mathbb{C}^n \longrightarrow \mathbb{E}^{s,u,c}$$

as the eigen-projection onto $\mathbb{E}^{s,u,c}$, that is, $\mathbb{P}^{s,u,c}V = V_{s,u,c}$. Here, we use the Euclidean norm for vectors on $\mathbb{C}^n$ and for matrices we use the operator norm induced by the Euclidean norm.

1.2. Main theorem

We now state the main theorem of this paper.

THEOREM 1.3. $\lambda \in \sigma_{L^2(\mathbb{R})}(L)$ *if and only if $R(\lambda)$ is not hyperbolic, that is, there exists a pure imaginary eigenvalue of $R(\lambda)$.*

REMARK 1.4. Here, the matrix $R(\lambda)$ is referred to as the Floquet matrix. By the above theorem, one can say that $\lambda \in \sigma_{L^2(\mathbb{R})}(L)$ if and only if there is a solution U of the ODE system (1.4) has the form

$$U(x) = e^{i\xi x}W(i\xi, x) \quad \text{for some } \xi \in \mathbb{R},$$

where $i\xi$ is an eigenvalue of $R(\lambda)$ and $W(i\xi,x) \in \mathbb{C}^{mn}$ is π-periodic in x. Since $U = (u, u', \cdots, u^{(n-2)}, Du^{(n-1)})^t$, the eigenfunction u of the eigenvalue problem (1.3) corresponding to $\lambda \in \sigma_{L^2(\mathbb{R})}(L)$ has the form

$$u(x) = e^{i\xi x}w(i\xi, x),$$

where w is the first m component of W. Since w is nonzero periodic in x, $u \notin L^2(\mathbb{R})$; so λ is not a point spectrum. Thus we conclude that $L^2(\mathbb{R})$-spectrum of L is entirely essential.

In order to prove Theorem 1.3, we first prove the following lemma, showing an exponential dichotomy of the system (1.4) when $R(\lambda)$ has no pure imaginary eigenvalue.

2. Exponential dichotomy

LEMMA 2.1. *For fixed $\lambda \in \mathbb{C}$, assume $R(\lambda)$ is hyperbolic, that is, $\mathbb{E}^c$ contains only zero vector. Then the system* (1.4) *has an exponential dichotomy, that is, for any vector $V \in \mathbb{C}^{mn}$, there are constants $M > 0$ and $k > 0$ such that*

$$|\Psi(x,\lambda)\mathbb{P}^s V| \le Me^{-kx}|V|, \quad x > 0,$$
$$|\Psi(x,\lambda)\mathbb{P}^u V| \le Me^{kx}|V|, \quad x < 0.$$

Thus, we say that $\Psi(x,\lambda)\mathbb{P}^s$ and $\Psi(x,\lambda)\mathbb{P}^u$ decay exponentially as $x \to \infty$ and $x \to -\infty$, respectively.

Proof. Recalling $R(\lambda)$ is a constant matrix, we denote the maximum real part of stable eigenvalues and the minimum real part of unstable eigenvalues of $R(\lambda)$ by α_M^s and α_m^u, respectively. Then we have $\alpha_M^s < 0 < \alpha_m^u$.

By using a Jordan canonical form, it is trivial that each entry of $e^{R(\lambda)x}$ is composed of linear combinations of $q(x)e^{\alpha x}\cos\beta x$ and $q(x)e^{\alpha x}\sin\beta x$, where α and β are real and complex parts of eigenvalues of $R(\lambda)$ and $q(x)$ is a polynomial with $deg(q(x)) \le mn-1$. We notice that if $\alpha < 0$ then $\alpha < \alpha_M^s < 0$ and if $0 < \alpha$ then $0 < \alpha_m^u < \alpha$.

Since $e^{R(\lambda)x}\mathbb{E}^s \subset \mathbb{E}^s$, for any $V \in \mathbb{C}^{mn}$ every entry of $e^{R(\lambda)x}\mathbb{P}^s V$ is composed of linear combinations of $q(x)e^{\alpha x}\cos\beta x$ and $q(x)e^{\alpha x}\sin\beta x$ with $\alpha < 0$; so we conclude that for all $\varepsilon > 0$, there is $M = M(\varepsilon) > 0$ such that

$$|P(x,\lambda)e^{R(\lambda)x}\mathbb{P}^s V| \le Me^{(\alpha_M^s+\varepsilon)x}|V|, \quad \text{for } x > 0.$$

Similarly, since $e^{R(\lambda)x}\mathbb{E}^u \subset \mathbb{E}^u$, we have

$$|P(x,\lambda)e^{R(\lambda)x}\mathbb{P}^u V| \le Me^{(\alpha_m^u+\varepsilon)x}|V|, \quad \text{for } x < 0.$$

Here, we used the fact that P is a continuous periodic function on $\mathbb{R}$ and ε controls the polynomials $q(x)$ because for each $\varepsilon > 0$ and $n \in \mathbb{N}$, $x^n \le \dfrac{n!}{\varepsilon^n}e^{\varepsilon x}$ for all $x \ge 0$. If $R(\lambda)$ is semisimple, then $q(x) \equiv 1$; so in this case we take $\varepsilon = 0$ and hence $k = \min\{|\alpha_M^s|, \alpha_m^u\}$. If $R(\lambda)$ is not semisimple, we take $\varepsilon > 0$ such that $\alpha_M^s + \varepsilon < 0$; hence we complete the proof with $k = \min\{|\alpha_M^s + \varepsilon|, \alpha_m^u + \varepsilon\}$. □

We are ready to prove Theorem 1.3.

3. Proof of the main theorem

Proof of Theorem 1.3. For fixed $\lambda \in \mathbb{C}$, we first assume $R(\lambda)$ is hyperbolic, that is, there is no pure imaginary eigenvalue of $R(\lambda)$ so that $\mathbb{E}^s \oplus \mathbb{E}^u = \mathbb{C}^{mn}$. In order to prove that λ lies in resolvent set of L, for any fixed $f \in L^2(\mathbb{R})$ there exists $u \in H^n(\mathbb{R})$ satisfying the following nonhomogeneous problem:

$$(L - \lambda)u = f. \tag{3.1}$$

We first rewrite (3.1) as the following nonhomogeneous ODE system

$$U' = \mathbb{A}(x, \lambda)U + F, \tag{3.2}$$

where U and $\mathbb{A}$ are introduced in (1.5) and $F = (0, 0, \cdots, 0, f)^t \in \mathbb{R}^{mn}$. Since $R(\lambda)$ is hyperbolic, $\mathbb{P}^s + \mathbb{P}^u = I_{mn}$; so by introducing the Green's function,

$$G(z) = \begin{cases} -\Psi(z, \lambda)\mathbb{P}^u(\lambda), & z < 0; \\ \Psi(z, \lambda)\mathbb{P}^s(\lambda), & z > 0, \end{cases} \tag{3.3}$$

we can solve (3.2) as

$$\begin{aligned} U(x) &= \int_{-\infty}^{\infty} G(x - y)F(y)dy \\ &= \int_{-\infty}^{x} \Psi(x - y, \lambda)\mathbb{P}^s F(y)dy - \int_{x}^{\infty} \Psi(x - y, \lambda)\mathbb{P}^u F(y)dy. \end{aligned} \tag{3.4}$$

Indeed, since $\Psi' = \mathbb{A}\Psi$ and $\Psi(0, \lambda) = I_{mn}$, by differentiating (3.4) we have

$$\begin{aligned} U'(x) = \mathbb{A} &\int_{-\infty}^{x} \Psi(x - y)\mathbb{P}^s F(y)dy \\ &- \mathbb{A} \int_{x}^{\infty} \Psi(x - y, \lambda)\mathbb{P}^u F(y)dy + (\mathbb{P}^s + \mathbb{P}^u)F(y) \\ = \mathbb{A}U(x) &+ F(y). \end{aligned}$$

Here, we used $\mathbb{P}^s + \mathbb{P}^u = I_{mn}$ because $R(\lambda)$ is hyperbolic. We now apply Young's inequality to (3.4), then we obtain

$$\|U\|_{L^2(\mathbb{R})} = \|G * F\|_{L^2(\mathbb{R})} \le \|G\|_{L^1(\mathbb{R})}\|F\|_{L^2(\mathbb{R})}.$$

We proved that the system (1.4) has an exponential dichotomy in Lemma 2.1, that is, G decays exponentially as $x \to \pm\infty$; so $\|G\|_{L^1(\mathbb{R})}$ is definitely bounded and hence

$$\|U\|_{L^2(\mathbb{R})} \le C\|F\|_{L^2(\mathbb{R})}.$$

Noting that $U = (u, \partial_x u, \cdots, \partial_x^{n-2}u, D\partial_x^{n-1}u)^t$ is continuous and D is nonsingular, we conclude that

$$\|u\|_{H^n(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})};$$

which means λ lies in the resolvent set.

We now prove the opposite direction by assuming $R(\lambda)$ is not hyperbolic which means there is an eigenvalue $i\xi$ of $R(\lambda)$ for some $\xi \in \mathbb{R}$. As a consequence of Floquet theory, there is a solution to the system (1.4) of the form

$$U(x, \lambda) = e^{i\xi x}p(x, \xi, \lambda)$$

where $p(x, \xi, \lambda) \in \mathbb{C}^{mn}$ is a π-periodic function in x. If $p_1 \in \mathbb{C}^m$ denotes the first m component of p, by recalling $u \in \mathbb{C}^m$ is the first m component of U, we know that $u(x) = e^{i\xi x}p_1(x, \xi, \lambda)$ solves $(L - \lambda)u = 0$ and $\|u\|_{L^2(\mathbb{R})} = \|p_1\|_{L^2(\mathbb{R})} = \infty$ because p_1 is a nonzero periodic function.

For each $k \in \mathbb{N}$, we now define $u_k(x) \in H^n(\mathbb{R})$ with $\|u_k\|_{H^n(\mathbb{R})} = 1$;

$$u_k(x) = C_k\chi_k u(x) = C_k\chi_k e^{i\xi x}p_1(x, \xi, \lambda),$$

where χ_k is a smooth cut off function, namely, $\chi_k(x) = 1$ for $|x| \le k$ and $\chi_k(x) = 0$ for $|x| \ge k+1$ such that $|\partial_x^j\chi(x)|$ is uniformly bounded for all $j = 0, 1, \cdots, n$, and C_k is a nonzero constant to make $\|u_k\|_{H^n(\mathbb{R})} = 1$, i.e., $C_k = \frac{1}{\|\chi_k p_1\|_{H^n(\mathbb{R})}}$. We then notice that $\lim_{k\to\infty} C_k = 0$ because $\lim_{k\to\infty} \chi_k(x) \equiv 1$ and $\lim_{k\to\infty} \|\chi_k p_1\|_{H^n(\mathbb{R})} = \infty$; which implies that $\lim_{k\to\infty} \|u_k\|_{W^{n,\infty}} = 0$.

In order to argue by contradiction we assume λ lies in the resolvent set of L. Then since $u_k(x) \in H^n(\mathbb{R})$, there is a constant $C > 0$, independent of k, such that

$$1 = \|u_k(x)\|_{H^n(\mathbb{R})} \le C\|(L - \lambda)u_k\|_{L^2(\mathbb{R})} \quad \text{for all } k \in \mathbb{N}.$$

However, we know that

$$\|(L - \lambda)u_k\|_{L^2(\mathbb{R})} = \|(L - \lambda)u_k\|_{L^2([-k-1,-k]\cup[k,k+1])} \le C\|u_k\|_{W^{n,\infty}};$$

which is a definitely contradiction to the fact that $\|u_k\|_{H^n(\mathbb{R})} = 1$ and $\lim_{k\to\infty} \|u_k\|_{W^{n,\infty}} = 0$. Thus λ lies in $L^2(\mathbb{R})$-spectrum of L. We complete the proof. □

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