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BOUNDEDNESS IN FUNCTIONAL PERTURBED DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

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Abstract. This paper shows that the solutions to the perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t)),$$

have bounded properties by imposing conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), T_1 y(s))ds$, $h(t, y(t), T_2 y(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t, y)$ using the notion of h-stability.

1. Introduction and preliminaries

The papers [2-6,8-11,14-17] discuss boundedness, perturbations, stability, and h-stability of nonlinear systems of differential equations,

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{1.1}$$

It is interesting and worthwhile to investigate the bounded perporty for the solutions of the perturbed type of (1.1)

(1.2)

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t)),\ y(t_0) = y_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g, h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$, $f(t, 0) = 0$, $g(t, 0, 0) = h(t, 0, 0) = 0$, and $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ are continuous operators and $\mathbb{R}^n$ is an n-dimensional Euclidean space.

The notion of h-stability (hS) was introduced by Pinto [16, 17] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability)

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under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Pachpatte [14, 15] investigated the stability, boundedness, and the asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term g and on the operator T. Choi and Ryu [6] and Choi *et al.* [7] investigated h-stability of solutions for nonlinear perturbed systems. Also, Goo [9, 10] and Goo *et al.* [3] studied the boundedness of solutions for the perturbed differential systems.

We always assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$. The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$. Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then, we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \;\; v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \;\; z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We recall some notions of h-stability [17].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 1.2. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called *h-stable* (hS) if there exists $\delta > 0$ such that (1.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [8].

DEFINITION 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, that is,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [8, 12].

Before proceeding to the statement of main results, we set forth some known results.

LEMMA 1.4. [17] *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \ y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 1.5. [2] *Let x and y be a solution of (1.1) and (1.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

THEOREM 1.6. [6] *If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.*

THEOREM 1.7. [7] *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS, then the solution $z = 0$ of (1.4) is hS.*

LEMMA 1.8. (*Bihari – type inequality*) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that for some $c > 0$*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big], \qquad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 1.9. [4] *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10} \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds \\ &+ \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \Big(\lambda_4(\tau) u(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r) u(r) dr \\ &+ \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r) w(u(r)) dr\Big) d\tau ds \\ &+ \int_{t_0}^{t} \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau) w(u(\tau)) d\tau ds. \end{aligned}$$

Then

$$\begin{aligned} u(t) \leq W^{-1}\Big[W(c) &+ \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) \\ &+ \lambda_3(s) \int_{t_0}^{s} (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r) dr \\ &+ \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r) dr) d\tau + \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau) d\tau\Big) ds\Big], \end{aligned}$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : \quad W(c) + \textstyle\int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^s(\lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^\tau \lambda_6(r)dr + \lambda_7(\tau)\int_{t_0}^\tau \lambda_8(r)dr)d\tau + \lambda_9(s)\int_{t_0}^s \lambda_{10}(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

We need the following two corollaries for the proof.

COROLLARY 1.10. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s)w(u(s))ds \\ &+ \int_{t_0}^t \lambda_3(s)\int_{t_0}^s \Big(\lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)u(r)dr \\ &+ \lambda_6(\tau)\int_{t_0}^\tau \lambda_7(r)w(u(r))dr\Big)d\tau ds \\ &+ \int_{t_0}^t \lambda_8(s)\int_{t_0}^s \lambda_9(\tau)w(u(\tau))d\tau ds. \end{aligned}$$

Then

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^s(\lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)dr \\ + \lambda_6(\tau)\int_{t_0}^\tau \lambda_7(r)dr)d\tau + \lambda_8(s)\int_{t_0}^s \lambda_9(\tau)d\tau\Big)ds\Big], \end{aligned}$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^s(\lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)dr + \lambda_6(\tau)\int_{t_0}^\tau \lambda_7(r)dr)d\tau + \lambda_8(s)\int_{t_0}^s \lambda_9(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

COROLLARY 1.11. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$,

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s)w(u(s))ds \\ &+ \int_{t_0}^t \lambda_3(s)\int_{t_0}^s \lambda_4(\tau)u(\tau)d\tau ds \\ &+ \int_{t_0}^t \lambda_5(s)\int_{t_0}^s \lambda_6(\tau)w(u(\tau))d\tau ds, \;\; 0 \le t_0 \le t. \end{aligned}$$

Then

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) &+ \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^s \lambda_4(\tau)d\tau \\ &+ \lambda_5(s)\int_{t_0}^s \lambda_6(\tau)d\tau)ds\Big], \end{aligned}$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : \; W(c) &+ \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^s \lambda_4(\tau)d\tau \\ &+ \lambda_5(s)\int_{t_0}^s \lambda_6(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

2. Main Results

In this section, we investigate boundedness for solutions of the perturbed differential systems via t_∞-similarity.

To obtain the bounded result for solutions of the perturbed differential systems, the following assumptions are needed:

(H1) $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.
(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h.
(H3) $w(u)$ is nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$.

THEOREM 2.1. *Let* $a, b, c, d, k, m, n, p, q \in C(\mathbb{R}^+)$. *Suppose that (H1), (H2), (H3), and* g *in (1.2) satisfies*

$$|g(t, y, T_1 y)| \le a(t)w(|y(t)|) + |T_1 y(t)|, \tag{2.1}$$

$$|T_1y(t)| \le b(t)\int_{t_0}^t k(s)|y(s)|ds + d(t)\int_{t_0}^t p(s)w(|y(s)|)ds, \tag{2.2}$$

and

$$\begin{aligned} &|h(t,y(t),T_2y(t))| \le c(t)w(|y(t)|) + |T_2y(t)|,\\ &|T_2y(t)| \le m(t)|y(t)| + n(t)\int_{t_0}^t q(s)|y(s)|ds, \end{aligned} \tag{2.3}$$

where $a,b,c,d,k,m,n,p,q \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* T_1, T_2 *are continuous operators. Then any solution* $y(t) = y(t,t_0,y_0)$ *of (1.2) is bounded on on* $[t_0,\infty)$ *and it satisfies*

$$\begin{aligned} |y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t \Big(c(s) + m(s) + \int_{t_0}^s (a(\tau)\\ +b(\tau)\int_{t_0}^\tau k(r)dr + d(\tau)\int_{t_0}^\tau p(r)dr)d\tau + n(s)\int_{t_0}^s q(\tau)d\tau\Big)ds\Big], \end{aligned}$$

where $t_0 \le t < b_1$, $c = c_1|y_0|\,h(t_0)^{-1}$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^t \Big(c(s) + m(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr\\ +d(\tau)\int_{t_0}^\tau p(r)dr)d\tau + n(s)\int_{t_0}^s q(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, from (H1), by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Applying the nonlinear variation of constants formula due to Lemma 1.5, together with (2.1), (2.2), and (2.3), we have

$$\begin{aligned} &|y(t)|\\ &\le |x(t)| + \int_{t_0}^t |\Phi(t,s,y(s))|\Big(\int_{t_0}^s |g(\tau,y(\tau),T_1y(\tau))|d\tau + |h(s,y(s),T_2y(s))|\Big)ds\\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1}\Big(\int_{t_0}^s (a(\tau)w(|y(\tau)|)\\ &\quad + b(\tau)\int_{t_0}^\tau k(r)|y(r)|dr + d(\tau)\int_{t_0}^\tau p(r)w(|y(r)|)dr)d\tau\\ &\quad + m(s)|y(s)| + c(s)w(|y(s)|) + n(s)\int_{t_0}^s q(\tau)|y(\tau)|d\tau\Big)ds. \end{aligned}$$

By the assumptions (H2) and (H3), we obtain

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2 h(t)\Big(m(s)\frac{|y(s)|}{h(s)} + c(s)w(\frac{|y(s)|}{h(s)})$$
$$+\int_{t_0}^s (a(\tau)w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^\tau k(r)\frac{|y(r)|}{h(r)}dr$$
$$+d(\tau)\int_{t_0}^\tau p(r)w(\frac{|y(r)|}{h(r)})dr)d\tau + n(s)\int_{t_0}^s q(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\Big)ds.$$

Let $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 1.9, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t \Big(c(s) + m(s) + \int_{t_0}^s (a(\tau)$$
$$+b(\tau)\int_{t_0}^\tau k(r)dr + d(\tau)\int_{t_0}^\tau p(r)dr)d\tau + n(s)\int_{t_0}^s q(\tau)d\tau\Big)ds\Big]$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the proof is complete. □

Remark 2.2. Letting $c(t) = k(t) = m(t) = q(t) = 0$ in Theorem 2.1, we obtain the similar result as that of Theorem 3.4 in [5].

Theorem 2.3. *Let $a, b, c, d, k, m, p, q \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$\int_{t_0}^t |g(s, y(s), T_1y(s))|ds \leq a(t)w(|y(t)|) + |T_1y(t)|, \tag{2.4}$$

$$|T_1y(t)| \leq b(t)\int_{t_0}^t k(s)|y(s)|ds + d(t)\int_{t_0}^t p(s)w(|y(s)|)ds, \tag{2.5}$$

$$|h(t, y(t), T_2y(t))| \leq b(t)\int_{t_0}^t c(s)|y(s)|ds + |T_2y(t)|, \tag{2.6}$$

and

$$|T_2y(t)| \leq m(t)|y(t)| + d(t)\int_{t_0}^t q(s)w(|y(s)|)ds, \tag{2.7}$$

where $a, b, c, d, k, m, p, q \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are continuous operators. Then any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} \Big(a(s) + m(s)$$
$$+b(s)\int_{t_0}^{s}(c(\tau)+k(\tau))d\tau + d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))d\tau\Big)ds\Big],$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + c_2\int_{t_0}^{t}(a(s)+m(s)+b(s)\int_{t_0}^{s}(c(\tau)+k(\tau))d\tau$$
$$+ d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))d\tau]ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula due to Lemma 1.5, together with (2.4),(2.5), (2.6), and (2.7), we have

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(a(s)w(|y(s)|)$$
$$+ b(s)\int_{t_0}^{s}(c(\tau)+k(\tau))|y(\tau)|d\tau$$
$$+ d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))w(|y(\tau)|)d\tau + m(s)|y(s)|\Big)ds.$$

It follows from (H2) and (H3) that

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(m(s)\frac{|y(s)|}{h(s)} + a(s)w(\frac{|y(s)|}{h(s)})$$
$$+b(s)\int_{t_0}^{s}(c(\tau)+k(\tau))\frac{|y(\tau)|}{h(\tau)}d\tau$$
$$+d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.$$

Let $u(t) = |y(t)||h(t)|^{-1}$. Then, by Corollary 1.11, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\Big(a(s)+m(s)$$
$$+ b(s)\int_{t_0}^{s}(c(\tau)+k(\tau))d\tau + d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))d\tau\Big)ds\Big],$$

where $c = c_1|y_0|h(t)\, h(t_0)^{-1}$. Thus any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

REMARK 2.4. Letting $c(t) = k(t) = m(t) = q(t) = 0$ in Theorem 2.3, we obtain the same result as that of Theorem 3.3 in [5].

THEOREM 2.5. *Let $a, b, c, d, k, m, p, q \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$|g(t, y, T_1 y)| \le a(t)w(|y(t)|) + |T_1 y(t)|, \tag{2.8}$$

$$|T_1 y(t)| \le b(t) \int_{t_0}^{t} k(s)|y(s)|ds + c(t) \int_{t_0}^{t} p(s)w(|y(s)|)ds, \tag{2.9}$$

and

$$\begin{aligned} h(t, y(t), T_2 y(t))| &\le \int_{t_0}^{t} q(s)w(|y(s)|)ds + |T_2 y(t)|, \\ |T_2 y(t)| &\le m(t)|y(t)| + d(t)w(|y(t)|), \end{aligned} \tag{2.10}$$

where $a, b, c, d, k, m, p, q \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T_1, T_2 are continuous operators. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$\begin{aligned} |y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} \Big(d(s) + m(s) + \int_{t_0}^{s} (a(\tau) + q(\tau) \\ + b(\tau) \int_{t_0}^{\tau} k(r)dr + c(\tau) \int_{t_0}^{\tau} p(r)dr)d\tau\Big)ds\Big], \end{aligned}$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} \Big(d(s) + m(s) + \int_{t_0}^{s} (a(\tau) + q(\tau) \\ + b(\tau) \int_{t_0}^{\tau} k(r)dr + c(\tau) \int_{t_0}^{\tau} p(r)dr)d\tau\Big)ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Applying the nonlinear variation of constants formula due to Lemma 1.5, together with (2.8), (2.9), and (2.10), we have

$$|y(t)| \\ \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(m(s)|y(s)| + d(s)w(|y(s)|) \\ + \int_{t_0}^{s}((a(\tau)+q(\tau))w(|y(\tau)|) + b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr \\ + c(\tau)\int_{t_0}^{\tau} p(r)w(|y(r)|)dr)d\tau\Big)ds.$$

By the assumptions (H2) and (H3), we obtain

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(m(s)\frac{|y(s)|}{h(s)} + d(s)w(\frac{|y(s)|}{h(s)}) \\ + \int_{t_0}^{s}((a(\tau)+q(\tau))w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^{\tau} k(r)\frac{|y(r)|}{h(r)}dr \\ + c(\tau)\int_{t_0}^{\tau} p(r)w(\frac{|y(r)|}{h(r)})dr)d\tau\Big)ds.$$

Let $u(t) = |y(t)||h(t)|^{-1}$. Then, by Corollary 1.10, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\Big(d(s) + m(s) + \int_{t_0}^{s}(a(\tau)+q(\tau) \\ + b(\tau)\int_{t_0}^{\tau} k(r)dr + c(\tau)\int_{t_0}^{\tau} p(r)dr)d\tau\Big)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded. This completes the proof. □

REMARK 2.6. Letting $d(t) = k(t) = m(t) = q(t) = 0$ in Theorem 2.5, we obtain the similar result as that of Theorem 3.4 in [5].

THEOREM 2.7. *Let $a, b, c, d, k, m, p, q \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s), T_1y(s))|ds \leq a(t)w(|y(t)|) + |T_1y(t)|, \tag{2.11}$$

$$|T_1y(t)| \leq b(t)\int_{t_0}^{t} k(s)|y(s)|ds + c(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \tag{2.12}$$

and

$$(2.13)\qquad \begin{aligned}&|h(t,y(t),T_2y(t))| \le c(t)\int_{t_0}^t q(s)w(|y(s)|)ds + |T_2y(t)|,\\ &|T_2y(t)| \le m(t)|y(t)| + d(t)w(|y(t)|)\end{aligned}$$

where $a,b,c,d,k,m,p,q \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 *are continuous operators. Then, any solution* $y(t) = y(t,t_0,y_0)$ *of (1.2) is bounded on* $[t_0,\infty)$ *and it satisfies*

$$\begin{aligned}|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t \Big(a(s)+d(s)+m(s)\\ +b(s)\int_{t_0}^s k(\tau)d\tau + c(s)\int_{t_0}^s (p(\tau)+q(\tau))d\tau\Big)ds\Big],\end{aligned}$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$\begin{aligned}b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^t \Big(a(s)+d(s)+m(s)\\ +b(s)\int_{t_0}^s k(\tau)d\tau + c(s)\int_{t_0}^s (p(\tau)+q(\tau))d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.\end{aligned}$$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula due to Lemma 1.5, together with (2.11), (2.12), and (2.13), we have

$$\begin{aligned}|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1}\Big(m(s)|y(s)|\\ +(a(s)+d(s))w(|y(s)|) + b(s)\int_{t_0}^s k(\tau)|y(\tau)|d\tau\\ +c(s)\int_{t_0}^s (p(\tau)+q(\tau))w(|y(\tau)|)d\tau\Big)ds.\end{aligned}$$

It follows from (H2) and (H3) that

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(m(s)\frac{|y(s)|}{h(s)}$$
$$+(a(s)+d(s))w(\frac{|y(s)|}{h(s)}) + b(s)\int_{t_0}^{s} k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau$$
$$+c(s)\int_{t_0}^{s}(p(\tau)+q(\tau))w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.$$

Let $u(t) = |y(t)||h(t)|^{-1}$. Then, by Corollary 1.11, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\Big(a(s)+d(s)+m(s)$$
$$+b(s)\int_{t_0}^{s}k(\tau)d\tau + d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))d\tau\Big)ds\Big],$$

where $c = c_1|y_0|h(t)\,h(t_0)^{-1}$. Thus any solution $y(t) = y(t,t_0,y_0)$ of (1.2) is bounded on $[t_0,\infty)$. Hence the proof is complete. □

REMARK 2.8. Letting $d(t) = k(t) = m(t) = q(t) = 0$ in Theorem 2.7, we obtain the same result as that of Theorem 3.3 in [5].

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