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A NOTE ON THE CHARACTERIZATIONS OF THE GUMBEL DISTRIBUTION BASED ON LOWER RECORD VALUES

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(x) < 1$ for all x in $(-\infty, \infty)$.

In this paper, we obtain the characterizations of the Gumbel distribution by lower record values.

1. Introduction

Suppose that $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with a cumulative distribution function (cdf) $F(x)$ and a probability density function (pdf) $f(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper(lower) record value of $\{X_n, n \geq 1\}$ if $Y_j > (<) Y_{j-1}$ for $j > 1$. By the definition, X_1 is an upper as well as a lower record value. One can transform the upper records to lower records by replacing the original sequence of $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or (if $P(X_j > 0) = 1$ for all j) by $\{1/X_j, j \geq 1\}$. The indices at which the upper record values occur are given by record times $\{U(n), n > 0\}$, where $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}\}, n > 1$ and $U(n) = 1$. We will denote $L(n)$, $n \geq 1$, as the indices where lower record values occur.

Shawky and Bakoban (2011) obtained characterization that $F(x)$ has a Gumbel distribution if and only if $e^{-L_n} - e^{-L_m}$ and e^{L_m} are independent for $1 \leq m < n$. Also, Nadarajah, Teimouri, and Shih (2014)

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presented characterizations of the Weibull and uniform distributions via the ratio of two record statistics.

In this paper we investigate the characterizations of the Gumbel distributions by lower record values.

2. Main results

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a cdf $F(x)$ which is absolutely continuous with a pdf $f(x)$ and $F(x) < 1$ for all x in $(-\infty, \infty)$. Then $X_k, k \geq 1$ has a Gumbel distribution if and only if $X_{L(m)} - X_{L(n)}$ and $X_{L(n)}$ are independent for $1 \leq m < n$.*

Proof. If $X_k \in Gumbel$, then it can easily be shown that $X_{L(n)}$ and $X_{L(m)} - X_{L(n)}$ are independent for $1 \leq m < n$. So we have to prove the converse.

The joint pdf $f_{m,n}(x, y)$ of $X_{L(m)}$ and $X_{L(n)}$ can be written as

$$f_{m,n}(x,y) = \frac{\{H(x)\}^{m-1}}{\Gamma(m)} h(x) \frac{\{H(y) - H(x)\}^{n-m-1}}{\Gamma(n-m)} f(y)$$

where $H(x) = -lnF(x)$ and $h(x) = -\frac{d}{dx}H(x)$.

Let us use the transformation $U = X_{L(m)} - X_{L(n)}$ and $V = X_{L(n)}$. The Jacobian of the transformation is $J = 1$. Thus we can write the joint pdf $f_{U,V}(u, v)$ of U and V as

$$\begin{aligned} & f_{U,V}(u,v) \\ (2.1) \qquad & = \frac{\{H(u+v)\}^{m-1}}{\Gamma(m)} h(u+v) \frac{\{H(v) - H(u+v)\}^{n-m-1}}{\Gamma(n-m)} f(v), \end{aligned}$$

for all $u > 0$ and $-\infty < v < \infty$.

Here the marginal pdf $f_V(v)$ of V is given by

$$f_V(v) = \frac{\{H(v)\}^{n-1}}{\Gamma(n)} f(v), \; -\infty < v < \infty. \tag{2.2}$$

From (2.1) and (2.2), we get the conditional pdf of $f(u \mid X_{L(n)} = v)$ as

(2.3)

$$
\begin{aligned}
f(u \mid X_{L(n)} = v) \\
&= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left(\frac{H(u+v)}{H(v)} \right)^{m-1} \left(1 - \frac{H(u+v)}{H(v)} \right)^{n-m-1} \frac{h(u+v)}{H(v)} \\
&= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left(\frac{H(u+v)}{H(v)} \right)^{m-1} \left(1 - \frac{H(u+v)}{H(v)} \right)^{n-m-1} \frac{\partial}{\partial u} \left(\frac{H(u+v)}{H(v)} \right)
\end{aligned}
$$

where $\frac{\partial}{\partial u}\left(\frac{H(u+v)}{H(v)} \right) \neq 0$ for all $u > 0$ and $-\infty < v < \infty$.

Since $U = X_{L(m)} - X_{L(n)}$ and $V = X_{L(n)}$ are independent, by using the lemma of Ahasanullah(see p.48 in [2]), $\frac{\partial}{\partial v}\left(\frac{H(u+v)}{H(v)} \right) = 0$.

Thus we get

$$
\frac{H(u+v)}{H(v)} = G(u) \tag{2.4}
$$

where $G(u)$ is a function of u only.

By the theory of a functional equation(see [1]), the only continuous solution of (2.4) with the boundary conditions $H(0) = 1$ and $H(\infty) = 0$ is

$$
H(v) = e^{-v}, \quad -\infty < v < \infty.
$$

Thus we have $F(x) = e^{-e^{-x}}$ for all x in $(-\infty, \infty)$. This completes the proof. □

THEOREM 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a cdf $F(x)$ which is absolutely continuous with a pdf $f(x)$ and $F(x) < 1$ for all x in $(-\infty, \infty)$. Then $W = e^{X_{L(m)} - X_{L(n)}}$ has the beta distribution with shape parameter $n - m$ and m if and only if $F(x) = e^{-e^{-x}}$, $-\infty < x < \infty$.*

Proof. The joint pdf $f_{m,n}(x, y)$ of $X_{L(m)}$ and $X_{L(n)}$ can be written as

$$
f_{m,n}(x, y) = \frac{\{H(x)\}^{m-1}}{\Gamma(m)\Gamma(n-m)} h(x) \{H(y) - H(x)\}^{n-m-1} f(y) \tag{2.5}
$$

for $1 \leq m < n$, where $H(x) = -lnF(x)$ and $h(x) = -\frac{d}{dx}H(x)$.

Consider the functions $U = X_{L(m)} - X_{L(n)}$ and $V = X_{L(n)}$. The Jacobian of the transformation is $J = 1$. Thus we can write the joint pdf $f_{U,V}(u, v)$ of U and V as

$$f_{U,V}(u,v) = \frac{\{H(u+v)\}^{m-1}}{\Gamma(m)\Gamma(n-m)} h(u+v)\{H(v)-H(u+v)\}^{n-m-1} f(v) \tag{2.6}$$

for $u > 0$ and $-\infty < v < \infty$.

If $F(x) = e^{-e^{-x}}$ for all x in $(-\infty, \infty)$, then we get

$$f_{U,V}(u,v) = \frac{\left(e^{-(u+v)}\right)^{m-1}}{\Gamma(m)\Gamma(n-m)} e^{-(u+v)} \left(e^{-v} - e^{-(u+v)}\right)^{n-m-1} e^{-v} e^{-e^{-v}} \tag{2.7}$$

for $u > 0$ and $-\infty < v < \infty$.

Integrating (2.7) with respect to v, we have

$$\begin{aligned} f_U(u) &= \frac{\left(e^{-u}\right)^m \left(1-e^{-u}\right)^{n-m-1}}{\Gamma(m)\Gamma(n-m)} \int_{-\infty}^{\infty} \left(e^{-v}\right)^n e^{-e^{-v}} dv \\ &= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left(e^{-u}\right)^m \left(1-e^{-u}\right)^{n-m-1}, \ u > 0. \end{aligned} \tag{2.8}$$

Now consider the transformation $W = e^{-U}$. Then the cdf $F_W(w)$ of W is $F_W(w) = P(e^{-U} \le w) = P(U \ge -ln(w)) = 1 - F_U(-ln(w))$, and hence the pdf $f_W(w)$ of W can be written as

$$f_W(w) = \frac{1}{w} f_U(-ln(w)) = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} w^{m-1}(1-w)^{n-m-1} \tag{2.9}$$

for all $0 < w < 1$.

Now we will prove the sufficient condition. Let us use the transformation $U = X_{L(m)} - X_{L(n)}$ and $V = X_{L(n)}$. The Jacobian of the transformation is $J = 1$. Then we can write the pdf $f_U(u)$ of U as

$$\begin{aligned} f_U(u) = \int_{-\infty}^{\infty} & \frac{\{-lnF(u+v)\}^{m-1} f(u+v)}{\Gamma(m)\Gamma(n-m)F(u+v)} \\ & \cdot \{-lnF(v) + lnF(u+v)\}^{n-m-1} f(v) dv, \ u > 0. \end{aligned} \tag{2.10}$$

By setting $-lnF(u+v) = t$, we get

$$f_U(u) = \int_0^{\infty} \frac{t^{m-1}\{-lnF(F^{-1}(e^{-t})-u)-t\}^{n-m-1}}{\Gamma(m)\Gamma(n-m)} f(F^{-1}(e^{-t})-u) dt \tag{2.11}$$

for $u > 0$.

Now consider the transformation $W = e^{-U}$. Then we get the pdf $f_W(w)$ of W as

$$
\begin{aligned}
f_W(w) &= \frac{1}{w} f_U(-ln(w)) \\
&= \frac{1}{w}\int_0^{\infty} \frac{t^{m-1}\{-lnF(F^{-1}(e^{-t}) + ln(w)) - t\}^{n-m-1}}{\Gamma(m)\Gamma(n-m)} \\
&\qquad \cdot f(F^{-1}(e^{-t}) + ln(w))dt, \qquad 0 < w < 1.
\end{aligned} \tag{2.12}
$$

Note that we must have $F^{-1}(e^{-t}) = -ln(t)$, so that the integral in (2.12) must be evaluated to induce a beta pdf with shape parameter $n - m$ and m.

Thus we have

$$F(x) = e^{-e^{-x}}, \ -\infty < x < \infty. \tag{2.13}$$

This completes the proof. □

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