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ON EXISTENCE AND DISTRIBUTION OF CONJUGATE POINTS IN FINSLER GEOMETRY

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Abstract. In this paper, we shall study the existence and distribution of conjugate points in Finsler geometry from the viewpoint of the oscillation and Morse index theory.

1. Introduction

With the appearance of the classical Myers-Auslander theorem [3] on the compactness of a complete Finsler manifold under an appropriate Ricci curvature condition, an entire field of research rose to clarify the interplay between curvature, Jacobi fields and conjugate points. This relationship has been investigated by many authors, notably, Galloway [5], and, more recently for instance by Anastasiei [2] and Mastrolia *et al.* [8]. In particular, these latter have shown that the original Myers-Auslander problem can be shifted to the analysis of the solutions f of the linear differential equation $f'' + Ric \cdot f = 0$. On the other hand, the above linear differential equation has been the subject of an intensive independent research in the last century (for an account, see [9]), and the possibility of exploiting these available analytical results has highly improved the original conclusions of Myers and Auslander.

Theorem 1.1. *Let M be a complete Finsler manifold. If, for some point $p \in M$, every geodesic γ issuing from $p \in M$ has the property that*

$$\int_0^\infty t^k \cdot \mathrm{Ric}\left(\gamma'(t)\right) dt = \infty$$

for some $k < 1$, then M is compact and has finite fundamental group.

One of the important features of this result is that the Ricci curvature is not required to be bounded below. Galloway [5] gave an example to

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show that the curvature condition of Theorem 1.1 cannot be improved to allow $k = 1$. Setting $k = 0$ we can also consult [6], Corollary 2.6 for a different proof and a generalization.

Adapting the Riccati inequality, we are able to extend Theorem 1.1 to the case where the Ricci curvature is bounded from below by a negative curvature.

THEOREM 1.2. *Let M be a complete n-dimensional Finsler manifold satisfying* $\mathrm{Ric} \geq -(n-1)B^2$ *for some* $B \geq 0$. *Suppose that, for each geodesic γ, there exist $0 < a < b$ and $k \neq 1$ for which either*

$$\int_a^b t \cdot \mathrm{Ric}\left(\gamma'(t)\right) dt > B\Big\{b + a\frac{e^{2Ba}+1}{e^{2Ba}-1}\Big\} + \frac{1}{4}\ln\Big(\frac{b}{a}\Big) \tag{1.3}$$

or

$$\int_a^b t^k \cdot \mathrm{Ric}\left(\gamma'(t)\right) dt > B\Big\{b^k + a^k\frac{e^{2Ba}+1}{e^{2Ba}-1}\Big\} + \frac{k^2}{4(1-k)}\Big(a^{k-1} - b^{k-1}\Big) \tag{1.4}$$

holds (if $B = 0$, this has to be intended in a limit sense). Then M is compact and has a finite fundamental group.

For complete Finsler manifolds without conjugate points and with integral Ricci curvature, the author [6] asserted that the integral of the Ricci curvature on the unit tangent bundle is nonpositive. By the same argument, and with the use of a new criterion (Lemma 4.1) for conjugate points, we also have the distribution of conjugate points along geodesics.

THEOREM 1.5. *Let M be a complete n-dimensional Finsler manifold with a finite volume and Ricci curvature bounded above. Then*

$$\int_{SM} \mathrm{Ric}(v)\, d\mu \leq \pi(n-1)^{1/2}\sqrt{\sup\Big\{0, \mathrm{Ric}\Big\}}\int_{SM} \Psi(v)\, d\mu.$$

In the above theorem, SM is the unit tangent bundle with the induced Liouville measure $d\mu$ and $\Psi : SM \to [0, \infty]$ is defined by

$$\Psi(v) = \liminf_{l\to\infty} \frac{1}{l}\Big\{\text{the number of points conjugate to } \gamma_v(0) \text{ along } \gamma_v|_{[0,l]}\Big\}.$$

Our proof is obtained by modifying some points in the proof from [4] and by checking that some facts proved in [8] for Riemannian manifolds hold also for Finsler manifolds. In the first section, we shall briefly review the Morse index theorem following [4]. In the second section, we shall study the existence of conjugate points using oscillation theory of linear differential equation [8]. In the last section, we shall discuss the distribution of conjugate points by use of Morse index and Birkhoff's ergodic theorem.

2. Preliminaries

In this section, we shall recall some well-known facts about Finsler geometry. See [4] for more details. Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F : TM \to [0, \infty)$ which has the following properties;

- F is smooth on $TM \setminus \{0\}$;
- $F(t \cdot v) = t \cdot F(v)$ for all $t \in \mathbb{R}^+,\ v \in T_xM$;
- for each $v \in T_xM \setminus \{0\}$, the following quadratic g^v is an inner product in T_xM,

$$g^v(u, w) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(v + su + tw)\Big]\Big|_{s=t=0}.$$

A manifold M endowed with a Finsler structure will be called a Finsler manifold.

For a fixed $v \in T_xM$, let γ_v be the geodesic from x with $\gamma_v'(0) = v$. Along γ_v, we have a family of inner products $g^t = g^{\gamma_v'(t)}$ in $T_{\gamma_v(t)}M$. Define the *Riemann curvature* $R^t : T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$ by

$$R^t\big(u(t)\big) := R^{\gamma_v'(t)}\big(u(t)\big) := R\big(U(t), V(t)\big)V(t),$$

where $U(t) = \big(\hat{\gamma}_v(t); u(t)\big)$ and $V(t) = \big(\hat{\gamma}_v(t); \gamma_v(t)\big) \in \pi^*TM$. Then the Ricci curvature is defined by

$$\mathrm{Ric}(v) := \sum_{i=1}^{n} g^v\big(R_v(e_i), e_i\big), v \in T_xM,$$

where $\{e_i\}_{i=1}^n$ is a g^v-orthonormal basis for T_xM.

A vector field $J = J(t)$ along γ_v is called a *Jacobi field* if it satisfies

$$D_tD_tJ(t) + R^t(J(t)) = 0.$$

The *exponential map* $\exp_x : T_xM \to M$ is defined as usual, that is, $\exp_x(v) = \gamma_v(1)$. A vector field J_u along γ_v with $J_u(0) = 0$ and $D_tJ_u(0) = u$ is a Jacobi field if and only if $J_u(t) = d\exp_x|_{tv}tu$. We can see that $\exp_x$ is singular at $rv \in T_xM$ if and only if there is $u \in T_xM\setminus\{0\}$, such that the Jacobi field J_u satisfies $J_u(l) = 0$. In this case we call $\gamma_v(l)$ a *conjugate point* with respect to x. To study the conjugate points along geodesics, we introduce the notion of index form. Let $\gamma(t) : [0, l] \to M$ be a geodesic. For vector fields $U = U(t)$ and $V = V(t)$ along $\gamma(t)$, define

$$\mathcal{I}_\gamma(U, V) := \int_0^l \Big\{ g^t\big(D_t U(t), D_t V(t)\big) - g^t\big(R^t(U(t)), V(t)\big)\Big\}\, dt. \tag{2.1}$$

$\mathcal{I}_\gamma(U, V)$ is called the *index form* along γ. There exists a conjugate points $\gamma(l)$ to $\gamma(0)$ along $\gamma(t)$, $0 < t \le l$, precisely when there exists a vector filed V along γ, where V is not the zero field, for which $\mathcal{I}_\gamma(V, V) \le 0$. If the only conjugate point to $\gamma(0)$ along γ is $\gamma(l)$, then the vector field V for which $\mathcal{I}_\gamma(V, V) \le 0$ are precisely the Jacobi fields vanishing at 0 and l. We first need the following lemma.

LEMMA 2.2. *Let $\gamma : [0, l] \to M$ be a geodesic. For any piecewise $\mathcal{C}^\infty$ vector field $V(t) \neq 0$ along $\gamma(t)$ with $V(0) = 0 = V(l)$, we have*

$$\mathcal{I}_\gamma(V, V) \ge 0$$

and the equality holds if and only if V is Jacobi field along γ.

The technique we shall use for establishing compactness criteria is provided by the following lemma.

LEMMA 2.3. [1] *Suppose there is a point p such that every geodesic γ issuing from p (in other words $\gamma(0) = p$) contains a point conjugate to p along γ. Then M is compact.*

For a geodesic γ, the dimension of a maximal subspace of vector fields for which $\mathcal{I}_\gamma$ is negative define is called the *index* of γ and denoted by ind_γ. The celebrated Morse index theorem establishes a direct relationship between the index of a geodesic and the total number of conjugate points along this geodesic counted with multiplicities.

THEOREM 2.4 (Morse index theorem). *Let $\gamma(t) : [0, l] \to M$ be a geodesic on a Finsler manifold of dimension n and $\gamma(t_1), \cdots, \gamma(t_k)$ ($0 < t_1, \cdots, t_k < l$) the conjugate points of $\gamma(0)$ along $\gamma|_{(0,l)}$, which appear isolated. Let $\eta(t_j)$ ($j = 1, \cdots, k$) be the multiplicity of $\gamma(t_j)$. Then*

$$\begin{aligned}\sum_{j=1}^k \mathrm{ind}_{\gamma|_{[t_{j-1}, t_j]}} - k \cdot (n-1) \le \mathrm{ind}_{\gamma|_{[0,l]}} &= \sum_{j=1}^k \eta(t_j) \\ &\le \sum_{j=1}^k \mathrm{ind}_{\gamma|_{[t_{j-1}, t_j]}} + k \cdot (n-1).\end{aligned}$$

Let Z_0 denote the subset of SM consisting of vectors v which the geodesic γ_v gives rise to no conjugate point of $\gamma_v(0)$, and Z the set of v for which $\Psi(v) = 0$. Then we have $Z_0 \subsetneq Z$ and Z invariant with respect to the geodesic flow. However, by Theorem 2.4 (Morse index theorem), we have

$$\mu(Z_0) = \mu(Z).$$

3. Ricci curvature and conjugate points

In this section we prove Theorem 1.1 and 1.2. We consider the trace of the Riccati equation for self-adjoint. Since trace and derivative commute, we get

$$f''(t) + \mathrm{Ric}\left(\gamma'(t)\right) \cdot f(t) = 0. \tag{3.1}$$

The associated nonlinear Riccati first order ordinary differential equation has been a useful tool in oscillation theory and related comparison theory for the second order linear Jacobi equation. The core of the proof lies in the the relationship between the index form for geodesics and the corresponding the Jacobi equation. See [6] for similar exploitation of this relationship. We need the following lemma to prove theorems.

Lemma 3.2. *Suppose there is a point p such that each geodesic γ issuing from p the differential equation (3.1) has infinitely many zeros. Then M is compact.*

Proof. The proof we give combines some standard Mores index techniques (Lemma 2.2) together with Lemma 2.3. Thus, it suffices to establish the existence of a point conjugate to p along each geodesic γ.

Since by assumption (3.1) has infinitely many zeros there exists a nontrivial solution $\phi : [0, \infty) \to \mathbf{R}$ to (3.1) such that $\phi(t_1) = \phi(t_2) = 0$ with $0 \le t_1 < t_2$. Define the function $f : [0, t_2] \to \mathbf{R}$ as follows:

$$f(x) = \begin{cases} 0, & \text{if } 0 \le t \le t_1; \\ \phi(t), & \text{if } t_1 \le t \le t_2. \end{cases}$$

We proceed by contraction. Let $\{E_i(t)\}_{i=1}^{n-1}$ is a g^t-orthonormal vector fields along $\gamma|_{[0,t_2]}$ orthogonal to $\gamma'(t)$. For each $i = 1, \cdots, n-1$, define $X_i(t) = f(t) \cdot E_i(t)$. Then a straightforward computation shows

$$g^t\Big(D_t D_t X_i + R^t\big(X_i, \gamma'(t)\big)\gamma'(t), V_i\Big) = \Big(f'' + g^t\big(R^t(E_i(t), E_i(t))\big) f\Big) f \tag{3.3}$$

on each of two subintervals $[0, t_1]$ and $[t_1, t_2]$ on which f is smooth. Thus substitution of (3.3) into (2.1) gives

$$\mathcal{I}_\gamma(X_i, X_i) = -\int_0^{t_2} \Big(f'' + g^t\big(R^t(E_i(t), E_i(t))\big)f\Big) f\, dt,$$

where we have used that fact that $X_i(t)$ vanishes as $t = t_1$. Then we have upon summation

$$\begin{aligned} \sum_{i=1}^{n-1} \mathcal{I}_\gamma(X_i, X_i) &= -(n-1)\int_0^{t_2} \Big(f'' + \mathrm{Ric}\,\big(\gamma'(t)\big)f\Big) f\, dt \\ &= -(n-1)\int_{t_1}^{t_2} \Big(\phi'' + \mathrm{Ric}\,\big(\gamma'(t)\big)\phi\Big) \phi\, dt = 0 \end{aligned}$$

since ϕ is a solution to (3.1). Therefore, for some i, $\mathcal{I}_\gamma(X_i, X_i) \le 0$. By Lemma 2.2, we must have $\mathcal{I}_\gamma(X_i, X_i) \ > 0$ unless there is some point conjugate to x along $\gamma_{[0,t_2]}$. □

Next, we shall study the compactness of Finsler manifolds, that is, results on the existence of conjugate points along geodesics which do not provide information about the location of conjugate points. We shall use the following results in the oscillation theory of linear differential equations.

Theorem 3.4. [9] *Let $\gamma : [0, \infty) \to M$ be a geodesic. If for some $k < 1$,*

$$\int_0^\infty t^k \cdot \mathrm{Ric}\,\big(\gamma'(t)\big)\, dt = \infty,$$

then every solution of (3.1) has infinitely many zeros.

The oscillation theory of linear differential equations (Theorem 3.4) in conjunction with Lemma 3.2 yields Theorem 1.1 of the generalization of Ambrose theorem [1].

Proof of Theorem 1.1. The same procedure can also be applied to the universal covering $\widetilde{M} \to M$, showing that $\widetilde{M}$ is compact and thus that $\pi_1(M)$ is finite. □

Remark 3.5. In [7], Kupeli studied the differential equations (3.1) with the Ricci curvature is a nonnegative, and proved if $\int_t^\infty \mathrm{Ric}\,\big(\gamma'(t)\big)\, dt$ exists and $\liminf_{t\to\infty} t \cdot \int_t^\infty \cdot \mathrm{Ric}\,\big(\gamma'(t)\big)\, dt > 1/4$, then every solution of (3.1) has infinitely many zeros and hence M is compact.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2 . By Lemma 3.2, it is enough to prove that a solution of (3.1) has a first zero. Suppose by contradiction that $f > 0$ on $[0, \infty)$. Hence the function $h(t) = f'(t)/f(t)$ satisfies the differential equation

$$h'(t) + h^2(t) \leq \mathrm{Ric}\left(\gamma'(t)\right) \leq B^2.$$

We compare h with the general solution h_C of $h_C' + h_C^2 = B^2$ given by

$$h_C(t) = B\frac{e^{2Bt} + C}{e^{2Bt} - C}, C \geq 1, t > 0.$$

Then we have

$$h(t) \leq h_1(t) = B\cosh(Bt) = B\frac{e^{2Bt} + 1}{e^{2Bt} - 1}, \text{ for all } t > 0. \tag{3.6}$$

Now, consider the case $k \neq 1$, and choose any $0 < a < b$. Integration by parts and using the estimate on h we deduce

$$\begin{aligned}
&\int_a^b t^k \cdot \mathrm{Ric}\left(\gamma'(t)\right) dt \\
&\quad = \int_a^b t^k \cdot \big(- h'(t) - h^2(t)\big)\, dt \\
&\quad = \int_a^b \left\{ - \big(t^k h(t)\big)' - t^k \Big(h(t) - \frac{k}{2t}\Big)^2 + \frac{k^2}{4} t^{k-2} \right\} dt \\
&\quad \leq -b^k h(b) + a^k h(a) + \frac{k^2}{4(1-k)}\Big(a^{k-1} - b^{k-1}\Big) \\
&\quad \leq \big(b^k B + a^k h_1(a)\big) + \frac{k^2}{4(1-k)}\Big(a^{k-1} - b^{k-1}\Big),
\end{aligned}$$

contradicting assumption (1.4), as desired. The case $k = 1$ is analogous, and $B = 0$ follows by taking the limit as $B \to 0$. □

Remark 3.7. With a slight improvement of the above technique, one can give an upper bound for the diameter of M. Suppose $\operatorname{diam} M > 2D$. Then there exist a geodeic γ with $\gamma(0) = p$ such that γ is minimizing at least on $(0, D)$. In analogy (3.6), this fact and Riccati inequality for h to satisfy

$$-B\frac{e^{2B(D-t)} + 1}{e^{2B(D-t)} - 1} \leq h(t) \leq B\frac{e^{2Bt} + 1}{e^{2Bt} - 1}.$$

This estimate leads one to obtain integral condition on $\mathrm{Ric}\left(\gamma'(t)\right)$, in the spirt of (1.4). For instance one can prove that $\operatorname{diam} M \leq 2D$ provided that

$$\int_0^D t^2 \cdot \operatorname{Ric}\big(\gamma'(t)\big)\, dt > D.$$

4. Distribution of conjugate points

In this section we discuss the density of conjugate points along geodesics and show Theorem 1.5. A new criterion for existence of conjugate points is obtained by use of Morse theory.

LEMMA 4.1. *Let $\gamma : [0,l] \to M$ be a geodesic on a Finsler manifold of dimension n. If*

$$\int_0^l \operatorname{Ric}\big(\gamma'(t)\big)\, dt \geq \pi(n-1)^{1/2} \sqrt{\max_{t\in[0,l]} \Big\{0, \operatorname{Ric}\big(\gamma'(t)\big)\Big\}},$$

and $\operatorname{Ric}(\gamma'(t))$ is not identically zero, then $\gamma(0)$ has a conjugate point $\gamma(r)$ along γ for some r in $(0,l]$.

Proof. Suppose that

$$\int_0^l \operatorname{Ric}\big(\gamma'(t)\big)\, dt \geq \pi(n-1)^{1/2} \sqrt{\max_{t\in[0,l]} \Big\{0, \operatorname{Ric}\big(\gamma'(t)\big)\Big\}}$$

and $\operatorname{Ric}\big(\gamma'(t)\big)$ is not identically zero.

To show that $\gamma(0)$ has a conjugate point along γ, it suffices to find a vector field $W : [0,l] \to TM$ along γ, which is not identically zero and for which $\mathcal{I}_\gamma(W,W) \leq 0$. Define $h : [0,l] \to \mathbb{R}$ by

$$h(t) = \begin{cases} \sin\left(\frac{\pi t}{2\beta}\right), & \text{if } 0 \leq t \leq \beta; \\ 1, & \text{if } \beta \leq t \leq l-\beta; \\ \sin\left(\frac{\pi(l-t)}{2\beta}\right), & \text{otherwise.} \end{cases}$$

Let $E(t)$ be a parallel unit vector field along $\gamma(t)$ that is g^t-orthogonal to $\gamma'(t)$, and let $V(t) = h(t)\cdot E(t)$. Then

$$\begin{aligned} \mathcal{I}_\gamma(V,V) &= \int_0^l \Big\{ g^t\big(D_t V(t), D_t V(t)\big) - g^t\big(R^t(V(t)), V(t)\big)\Big\}\, dt \\ &= \int_0^\beta h'(t)^2 + (1-h(t)^2) g^t\big(R^t(E(t)), E(t)\big)\, dt \\ &\qquad\qquad + \int_{l-\beta}^l h'(t)^2 + (1-h(t)^2) g^t\big(R^t(E(t)), E(t)\big)\, dt \end{aligned}$$

$$-\int_0^l g^t\big(R^t(E(t)), E(t)\big)\, dt.$$

Let $E_1(t), \cdots, E_{n-1}(t)$ be mutually g^t-orthonormal parallel vector field along γ that is g^t-orthogonal to $\gamma'(t)$. Let $V_i(t) = h(t) \cdot E_i(t)$ for $i = 1, \cdots, n-1$. Then we have

$$\begin{aligned}\sum_{i=1}^{n-1} \mathcal{I}_\gamma(V_i, V_i) = & \int_0^\beta (n-1)h'(t)^2 + (1-h(t)^2)\,\mathrm{Ric}\big(\gamma'(t)\big)\, dt \\ & + \int_{l-\beta}^l (n-1)h'(t)^2 + (1-h(t)^2)\,\mathrm{Ric}\big(\gamma'(t)\big)\, dt \\ & - \int_0^l \mathrm{Ric}\big(\gamma'(t)\big)\, dt \\ & \leq \int_0^\beta (n-1)h'(t)^2 + (1-h(t)^2)\beta\, dt \\ & + \int_{l-\beta}^l (n-1)h'(t)^2 + (1-h(t)^2)\beta\, dt \\ & - x\pi(n-1)^{1/2}\sqrt{\max_{t\in[0,l]} \mathrm{Ric}\big(\gamma'(t)\big)} = 0,\end{aligned}$$

and hence by Lemma 3.2, $\gamma(0)$ has a conjugate points along γ. □

An immediate consequence of Lemma 4.1 is the following supplement to a result of the author [6].

Corollary 4.2. *Let $\gamma : [0,\infty) \to M$ be a geodesic on a Finsler manifold of diemesion n, which gives rise to no conjugate point of $\gamma(0)$. Then*

$$\limsup_{l\to\infty} \int_0^l \mathrm{Ric}\big(\gamma'(t)\big)\, dt \geq \pi(n-1)^{1/2}\sqrt{\max_{t\in[0,l]}\Big\{0, \mathrm{Ric}\big(\gamma'(t)\big)\Big\}}.$$

The author showed that with the same hypotheses

$$\lim_{l\to\infty} \int_0^l \mathrm{Ric}\big(\gamma'(t)\big)\, dt \neq \infty.$$

Lemma 4.1, with the help of Birkhoff's ergodic and Morse index theorem, gives Theorem 1.5.

Proof of Theorem 1.5 . Let $\varphi_t : SM \to SM$ be the geodesic flow. Then the geodesic flow preserves induced Liouville measure $d\mu$. Recall that $Z_0 = \{v \in SM$: the geodesic γ_v have no conjugate points of

$\gamma_v(0)\}$ and $Z = \{v \in SM : \Psi(v) = 0\}$. Then Z is invariant for φ_t and $\mu(Z_0) = \mu(Z)$. Letting $\alpha = \sup_{v\in SM} \mathrm{Ric}(v)$ we have that α is positive. Using the author's criterion [6] for no conjugate points,

$$\int_{Z_0} Ric(v)\, d\mu = \int_Z Ric(v)\, d\mu \le 0,$$

it suffices to prove

$$\int_{SM-Z_0} \mathrm{Ric}(v)\, d\mu \le \pi\sqrt{\alpha(n-1)} \int_{SM} \Psi(v)\, d\mu.$$

By Lemma 4.1, if $\gamma(t)$ has exactly $N\big(\gamma'(0), l\big)$ conjugate points along $\gamma|_{[0,l]}$, then

$$\int_0^l \mathrm{Ric}\,\big(\gamma'(t)\big)\, dt \le \big(N(\gamma'(0), l\big) + 1)\pi\sqrt{\alpha(n-1)}$$

so that

$$\frac{1}{l}\int_0^l \mathrm{Ric}\,\big(\gamma'(t)\big)\, dt \le \pi\sqrt{\alpha(n-1)} \cdot \frac{N\big(\gamma'(0), l\big) + 1}{l}. \tag{4.3}$$

By Birkhoff's ergodic theorem and (4.3), we have

$$\begin{aligned}
\int_{SM-Z_0} \mathrm{Ric}(v)\, d\mu &= \int_{SM-Z} \mathrm{Ric}(v)\, d\mu \\
&\le \int_{SM-Z} \left\{ \lim_{l\to\infty} \frac{1}{l}\int_0^l \mathrm{Ric}\,\big(\varphi_t(v)\big)\, dt \right\} d\mu \\
&= \int_{SM-Z} \left\{ \liminf_{l\to\infty} \frac{1}{l}\int_0^l \mathrm{Ric}\,\big(\gamma_v'(t)\big)\, dt \right\} d\mu \\
&\le \int_{SM-Z} \left\{ \liminf_{l\to\infty} \pi\sqrt{\alpha(n-1)} \cdot \frac{N(v,l)+1}{l} \right\} d\mu \\
&= \pi\sqrt{\alpha(n-1)} \int_{SM-Z} \liminf_{l\to\infty} \frac{N(v,l)+1}{l}\, d\mu.
\end{aligned}$$

Since the right hand side of the above last line is

$$\pi\sqrt{\alpha(n-1)} \int_{SM} \Psi(v)\, d\mu.$$

we have done. □

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