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A NOTE ON MULTIPLIERS OF *AC*-ALGEBRAS

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Abstract. In this paper, we introduce the notion of multiplier of AC-algebra and consider the properties of multipliers in AC-algebras. Also, we characterized the fixed set $Fix_d(X)$ by multipliers. Moreover, we prove that $M(X)$, the collection of all multipliers of AC-algebras, form a semigroup under certain binary operation.

1. Introduction

In [2], a partial multiplier on a commutative semigroup $(A, \cdot)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x) \cdot y = x \cdot F(y)$ for all $x, y \in D_F$. In this paper, we introduce the notion of multiplier of AC-algebra and consider the properties of multipliers in AC-algebras. Also, we characterized the fixed set $Fix_d(X)$ by multipliers. Moreover, we prove that $M(X)$, the collection of all multipliers of AC-algebras, form a semigroup under certain binary operation.

2. Preliminaries

An algebra $(X, *, 0)$ with a binary operation $*$ is called an AC-*algebra* if it satisfies the following axioms for all $x, y \in X$,

(A1) $x * (y * z) = (x * y) * z$,
(A2) $x * y = y * x$,
(A3) $x * y = 0$ if and only if $x = y$,

In an AC-algebra X, the following properties hold for all $x, y, z \in X$,

(A4) $(x * y) * z = (x * z) * y$,
(A5) $(x * (x * y)) * y = 0$,
(A6) $0 * (x * y) = (0 * x) * (0 * y)$,

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(A7) $((x * z) * (y * z)) * (x * y) = 0$,
(A8) $((x * y) * (x * z)) * (z * y) = 0$,
(A9) $x * y = 0$ if and only if $(x * z) * (y * z) = 0$,
(A10) $x * y = 0$ if and only if $(z * x) * (z * y) = 0$,
(A11) $x * y = x$ if and only if $y = 0$,
(A12) $x = y * (y * x)$,
(A13) $(x * (x * y)) * (x * y) = y * (y * x)$,
(A14) $x * x = 0$,
(A15) $x * 0 = x$.

A non-empty subset A of X is called a *subalgebra* of X if $x * y \in A$ for all $x, y \in A$.
Let X be a AC-algebra. We define the binary operation "$\leq$" as the following,

$$x \leq y \Leftrightarrow x * y = 0$$

for all $x, y \in X$.

DEFINITION 2.1. A non-empty subset I of X is called an *ideal* of X if

(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in I$.

LEMMA 2.2. *Let $(X, *, 0)$ be an AC-algebra. Then the following holds true.*

(i) *The left cancellation laws holds, i.e., $z * x = z * y$ implies $x = y$.*
(ii) *The right cancellation laws holds, i.e., $x * z = y * z$ implies $x = y$.*

Proof. (i) Let $z * x = z * y$ for all $x, y, z \in X$. Then $x = z * (z * x) = z * (z * y) = y$.

(ii) Let $x * z = y * z$ for all $x, y, z \in X$. Then $x = z * (z * x) = z * (x * z) = z * (y * z) = z * (z * y) = y$. □

For an AC-algebra, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

3. Multipliers of AC-algebras

In what follows, let X denote a AC-algebra unless otherwise specified.

DEFINITION 3.1. Let X be a AC-algebra. By a *multiplier* of X, we mean a self map f of X satisfying the identity

$$f(x * y) = f(x) * y$$

for all $x, y \in X$.

Example 3.2. *Let* $X := \{0, 1, 2, 3\}$ *be a set in which "$*$" is defined by*

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

It is easy to check that $(X, *)$ *is an* AC*-algebra. Define a map* $f : X \to X$ *by*

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \\ 2 & \text{if } x = 1 \\ 1 & \text{if } x = 2 \\ 0 & \text{if } x = 3. \end{cases}$$

Then it is easy to check that f *is a multiplier of an* AC*-algebra* X.

Example 3.3. *Let* $\mathbb{Z}$ *be the set of all integers and "* $-$ *" be a minus operation on* $\mathbb{Z}$. *Then* $(\mathbb{Z}, -, 0)$ *is a* AC*-algebra. Let* $f(x) = x - 1$ *for all* $x \in \mathbb{Z}$. *Then*

$$f(x - y) = (x - y) - 1 = (x - 1) - y = f(x) - y$$

for all $x, y \in \mathbb{Z}$, *and so* f *is a multiplier of* X.

Definition 3.4. A self map f of an AC-algebra X is said to be *regular* if $f(0) = 0$.

Proposition 3.5. *Let* f *be a multiplier of* X. *Then*

(i) $f(0) = f(x) * x$ *for all* $x \in X$.
(ii) f *is* $1 - 1$.

Proof. (i) Let $x \in X$. Then $x * x = 0$. Hence we have

$$f(0) = f(x * x) = f(x) * x$$

for all $x \in X$.

(ii) Let $x, y \in X$ be such that $f(x) = f(y)$. Then by (i), we have $f(0) = f(x) * x$ and $f(0) = f(y) * y$. Thus

$$f(x) * x = f(y) * y$$

which implies $f(x) * x = f(x) * y$. By Lemma 2.1, we have $x = y$. □

Theorem 3.6. *Let* f *be a multiplier of* X. *Then* $f(x) = x$ *if and only if* f *is regular.*

Proof. Let f is regular. Then we have

$$f(0) = f(x * x) = f(x) * x = 0,$$

which implies $f(x) = x$ from (A3). Conversely, let $f(x) = x$ for all $x \in X$. Then it is clear that $f(0) = 0$, which f is regular. □

THEOREM 3.7. *Let X be a AC-algebra. If f is a regular multiplier of X, then $f(x) \leq x$ for all $x, y \in X$.*

Proof. Let f be a regular multiplier of X. Then we have $f(0) = f(x * x) = f(x) * x = 0$, i.e., $f(x) \leq x$. □

PROPOSITION 3.8. *Let X be a AC-algebra and let f be a multiplier of X. If $f(x) * x = 0$ for all $x \in X$, then f is regular.*

Proof. Let $f(x) * x = 0$ and let f be a multiplier of X. Then $f(0) = f(x * x) = f(x) * x = 0$, which implies that f is a regular multiplier of X. □

PROPOSITION 3.9. *Let f be a multiplier of X. Then the following holds true.*

(i) *If there is an element $x \in X$ such that $f(x) = x$, then f is the identity.*
(ii) *If there is an element $x \in X$ such that $f(y) * x = 0$ or $x * f(y) = 0$ for all $y \in X$, then $f(y) = x$, i.e., f is constant.*

Proof. (i) Let $f(x) = x$ for some $x \in X$. Then $f(x) * x = 0$ by (A3). Hence $f(0) = 0$ from Proposition 3.5 (i), i.e., f is regular. This implies that f is an identity map by Theorem 3.6.
(ii) It follows directly from (A3). □

PROPOSITION 3.10. *Let X be an AC-algebra. Then every idempotent multiplier of X is an endomorphism on X.*

Proof. Let f be an idempotent multiplier of X. Then $f^2(x) = f(x)$ for all $x, y \in X$. Let $x, y \in X$. Then

$$\begin{aligned} f(x * y) &= f^2(x * y) = f(f(x * y)) \\ &= f(f(x) * y) = f(y * f(x)) \\ &= f(y) * f(x) = f(x) * f(y), \end{aligned}$$

which implies that f is an endomorphism on X. □

PROPOSITION 3.11. *Let X be a AC-algebra and f be a multiplier of X. Then $f(x * f(x)) = 0$ for all $x \in X$.*

Proof. Let $x \in X$. Then we have

$$f(x * f(x)) = f(x) * f(x) = 0.$$

This completes the proof. □

Proposition 3.12. *Let X be an AC-algebra and let f be a regular multiplier. Then $f : X \to X$ is an identity map if it satisfies $f(x) * y = x * f(y)$ for all $x, y \in X$*

Proof. Since f is regular, we have $f(0) = 0$. Let $x * f(y) = f(x) * y$ for all $x, y \in X$. Then $f(x) = f(x * 0) = f(x) * 0 = x * f(0) = x * 0 = x$. Thus f is an identity map. □

Definition 3.13. Let f be a multiplier of X. An ideal I of X is said to be f-*invariant* if $f(I) \subseteq I$.

Theorem 3.14. *Let f be a multiplier of X. Then f is regular if and only if every ideal of X is f-invariant.*

Proof. Let f be a regular multiplier of X. Then by Theorem 3.6, $f(x) = x$ for all $x \in X$. Now $y \in f(I)$ where I is an ideal of X. Then $y = f(x)$ for some $x \in I$. Thus $y * x = f(x) * x = x * x = 0 \in I$, which implies $y \in I$ and $f(I) \subset I$. This implies that I is f-invariant. Conversely, let every ideal of X be f-invariant. Then $f(\{0\}) \subset \{0\}$. Hence $f(0) = 0$, which implies that f is regular. □

Let f be a multiplier of X. Define a set $Fix_f(X)$ by

$$Fix_f(X) := \{x \in X \mid f(x) = x\}$$

for all $x \in X$.

Proposition 3.15. *Let f be a multiplier of X. Define*

$$f \circ f(x) = f(f(x))$$

for all $x \in X$. If $x \in Fix_f(X)$, then we have $f \circ f(x) = x$ for all $x \in X$.

Proof. Let $x \in Fix_f(X)$. Then we have

$$f \circ f(x) = f(f(x)) = f(x) = x.$$

This completes the proof. □

Proposition 3.16. *Let X be an AC-algebra and let f be a multiplier on X. If $y \in Fix_f(X)$, we have $x \wedge y \in Fix_f(X)$ for all $x \in X$.*

Proof. Let f be a multiplier of X and let $y \in Fix_f(X)$. Then we get for all $x \in X$,

$$\begin{aligned} f(x \wedge y) &= f(y * (y * x)) = f(y) * (y * x) \\ &= y * (y * x) = x \wedge y. \end{aligned}$$

This completes the proof. □

THEOREM 3.17. *Let f and g be two idempotent multipliers of X such that $f \circ g = g \circ f$. Then the following conditions are equivalent.*

(i) $f = g$.
(ii) $f(X) = g(X)$.
(iii) $Fix_f(X) = Fix_g(X)$.

Proof. (i) ⇒ (ii): It is obvious.

(ii) ⇒ (iii): Let $f(X) = g(X)$ and $x \in Fix_f(X)$. Then $x = f(x) \in f(X) = g(X)$. Hence $x = g(y)$ for some $y \in X$. Now $g(x) = g(g(y)) = g^2(y) = g(y) = x$. Thus $x \in Fix_g(X)$. Therefore, $Fix_f(X) \subseteq Fix_g(X)$. Similarly, we can obtain $Fix_g(X) \subseteq Fix_f(X)$. Thus $Fix_f(X) = Fix_g(X)$.

(iii) ⇒ (i): Let $Fix_f(X) = Fix_g(X)$ and $x \in X$. Since $f(x) \in Fix_f(X) = Fix_g(X)$, we have $g(f(x)) = f(x)$. Also, we obtain $g(x) \in Fix_g(X) = Fix_f(X)$. Hence we get $f(g(x)) = g(x)$. Thus we have

$$f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x).$$

Therefore, f and g are equal in the sense of mappings. □

Let X be an *AC*-algebra. Then, for each $a \in X$, we define a map $f_a : X \to X$ by

$$f_a(x) = x * a$$

for all $x \in X$.

THEOREM 3.18. *For each $a \in X$, the map f_a is a multiplier of X.*

Proof. Suppose that f_a is a map defined by $f_a(x) = x * a$ for each $x \in X$. Then for any $x, y \in X$, we have by (A4),

$$\begin{aligned} f_a(x * y) &= (x * y) * a = (x * a) * y \\ &= f_a(x) * y. \end{aligned}$$

Hence f_a is a multiplier of X. This completes the proof. □

We call the multiplier f_a of Theorem 3.14 as *simple multiplier*.

PROPOSITION 3.19. *Let X be an AC-algebra. Then $f_0(x) = x$ for all $x \in X$, i.e., f_0 is the identity map of X.*

Proof. Let $x \in X$. Then

$$f_0(x) = x * 0 = x.$$

Hence f_0 is the identity map of X. □

PROPOSITION 3.20. *For $p \in X$, the mapping $\beta_p(a) = (a * p) * p$ is a multiplier of X.*

Proof. Let $p \in X$. Then we have

$$\begin{aligned}\beta_p(a * b) &= ((a * b) * p) * p \\ &= ((a * p) * b) * p \\ &= ((a * p) * p) * b \\ &= \beta_p(a) * b\end{aligned}$$

for all $a, b \in X$. This completes the proof. □

Let X be a AC-algebra. Define $f_a \circ f_b$ by

$$f_a \circ f_b(x) = f_a(f_b(x) = f_a(x * b) = (x * b) * a$$

for all $x, y \in X$.

THEOREM 3.21. *The composition of two simple multipliers of an AC-algebra is a commutative and associative binary operation.*

Proof. If f_a, f_b and f_c are multipliers of an AC-algebra, then for all $x, y, z \in X$,

$$\begin{aligned}f_a \circ f_b(x) &= (x * b) * a = x * (b * a) = x * (a * b) \\ &= (x * a) * b = (f_b \circ f_a)(x)\end{aligned}$$

and

$$\begin{aligned}(f_a \circ f_b) \circ f_c(x) &= (f_a \circ f_b)(x * c) = ((x * c) * b) * a = ((x * c) * b) * a \\ &= f_a((x * c) * b) = f_a \circ (f_b \circ f_c)(x).\end{aligned}$$

This completes the proof. □

THEOREM 3.22. *Let X be an AC-algebra and $a, b \in X$. If $f_a \circ f_b(x) = f_0(x)$ for all $x \in X$, then $f_a(x) = f_b(x)$.*

Proof. Let X be an AC-algebra and $a, b \in X$. Then

$$f_a \circ f_b(x) = (x * b) * a = x * (b * a) = f_0(x) = x$$

for all $x \in X$. From (A11), we have $b * a = 0$. Hence by (A2) and (A3), we have $a = b$, which implies $f_a(x) = f_b$ for all $x \in X$. □

PROPOSITION 3.23. *Let X be an AC-algebra and let f_1, f_2 be two multipliers of X. Then $f_1 \circ f_2$ is also a multiplier of X.*

Proof. Let f_1, f_2 be multipliers of X and $x, y \in X$. Then

$$\begin{aligned} f_1 \circ f_2(x * y) &= f_1((f_2(x * y)) = f_1(f_2(x) * y) \\ &= f_1(f_2(x)) * y = f_1 \circ f_2(x) * y. \end{aligned}$$

This completes the proof. □

Let X_1 and X_2 be two AC-algebras. Then $X_1 \times X_2$ is also a AC-algebra with respect to the point-wise operation given by

$$(a, b) * (c, d) = (a * c, b * d)$$

for all $a, c \in X_1$ and $b, d \in X_2$.

PROPOSITION 3.24. *Let X_1 and X_2 be two AC-algebras with a zero element respectively. Define a map $f : X_1 \times X_2 \to X_1 \times X_2$ by $f(x, y) = (0, y)$ for all $(x, y) \in X_1 \times X_2$. If $0 * x = 0$ for all $x \in X$, then f is a multiplier of $X_1 \times X_2$ with respect to the point-wise operation.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. The we have

$$\begin{aligned} f((x_1, y_1) * (x_2, y_2)) &= f(x_1 * x_2, y_1 * y_2) \\ &= (0, y_1 * y_2) \\ &= (0 * x_2, y_1 * y_2) \\ &= (0, y_1) * (x_2, y_2) \\ &= f(x_1, y_1) * (x_2, y_2). \end{aligned}$$

Therefore f is a multiplier of the direct product $X_1 \times X_2$. □

DEFINITION 3.25. Let X be an AC-algebra and let f_1, f_2 be two maps of X. Define the binary operation $\wedge$ as

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$$

for all $x \in X$.

PROPOSITION 3.26. *Let X be an AC-algebra and let f_1, f_2 be two multipliers of X. Then $f_1 \wedge f_2$ is a multiplier of X.*

Proof. Let X be an AC-algebra and let f_1, f_2 be two multipliers of X. Then by (A12) we have

$$\begin{aligned} (f_1 \wedge f_2)(x * y) &= f_1(x * y) \wedge f_2(x * y) \\ &= (f_1(x) * y) \wedge (f_2(x) * y) \\ &= (f_2(x) * y) * [(f_2(x) * y) * (f_1(x) * y)] \\ &= f_1(x) * y. \end{aligned}$$

On the other hand, we get from (A12),

$$\begin{aligned}(f_1 \wedge f_2)(x) * y &= (f_1(x) \wedge f_2(x)) * y \\ &= (f_2(x) * (f_2(x) * f_1(x))) * y \\ &= f_1(x) * y.\end{aligned}$$

Hence we have $(f_1 \wedge f_2)(x * y) = (f_1 \wedge f_2)(x) * y$. □

THEOREM 3.27. *If X is an AC-algebra, $(M(X), \wedge)$ forms a semigroup where $M(X)$ denotes the set of all multipliers of X.*

Proof. Let $f_1, f_2, f_3 \in M(X)$. Then

$$\begin{aligned}((f_1 \wedge f_2) \wedge f_3)(x * y) &= (f_1 \wedge f_2)(x * y) \wedge f_3(x * y) \\ &= f_3(x * y) * (f_3(x * y) * (f_1 \wedge f_2)(x * y)) \\ &= (f_1 \wedge f_2)(x * y) \\ &= f_1(x * y) \wedge f_2(x * y).\end{aligned}$$

Also, we have

$$\begin{aligned}(f_1 \wedge (f_2 \wedge f_3))(x * y) &= (f_1(x * y)) \wedge (f_2 \wedge f_3)(x * y) \\ &= f_1(x * y) \wedge ((f_2(x * y) \wedge f_3(x * y)) \\ &= f_1(x * y) \wedge f_2(x * y).\end{aligned}$$

This shows that $(f_1 \wedge f_2) \wedge f_3 = f_1 \wedge (f_2 \wedge f_3)$. Thus $M(X)$ forms a semigroup. □

DEFINITION 3.28. A AC-algebra X is said to be *positive implicative* if

$$(x * y) * z = (x * z) * (y * z) \quad \text{for all } x, y, z \in X.$$

Let $M(X)$ denotes the collection of all multipliers on X. Obviously, $0 : X \to X$ defined by $0(x) = 0$ for all $x \in X$ and $1 : X \to X$ defined by $1(x) = x$ for all $x \in X$ are in $M(X)$. Hence $M(X)$ is non-empty.

DEFINITION 3.29. A AC-algebra X is said to be *positive implicative* if

$$(x * y) * z = (x * z) * (y * z) \quad \text{for all } x, y, z \in X.$$

DEFINITION 3.30. Let X be a AC-algebra and let $M(X)$ be the collection of all multipliers on X. We define a binary operation "$*$" on M(X) by

$$(f * g)(x) = f(x) * g(x) \quad \text{for all } x \in X \quad \text{and } f, g \in M(X).$$

THEOREM 3.31. *Let X be a positive implicative AC-algebra. Then $(M(X), *, 0)$ is a positive implicative AC-algebra of X.*

Proof. (i) Let X be a AC-algebra and let $f, g \in M(X)$. Then

$$\begin{aligned}(g * f)(x * y) &= (g(x * y)) * (f(x * y)) \\ &= (g(x) * y) * (f(x) * y) \\ &= (g(x) * f(x)) * y = ((g * f))(x) * y,\end{aligned}$$

which implies $g * f \in M(X)$.

(ii) Let $f, g \in M(X)$. Then $(f * g)(x) = f(x) * g(x) = g(x) * f(x) = (g * f)(x)$ for all $x \in X$. Hence $f * g = g * f$ for all $f, g \in M(X)$.

(iii) Let $f, g, h \in M(X)$. Then $(f * (g * h))(x) = (f(x) * (g(x) * h(x))) = (f(x) * g(x)) * h(x) = ((f * g) * h)(x)$ for all $x \in X$. Hence $f * (g * h) = (f * g) * h$.

(iv) Let $f * g = 0$ for all $f, g \in M(X)$. Then $f(x) * g(x) = 0$. Hence $f(x) = g(x)$, which implies $f = g$. Conversely, let $f = g$ for all $f, g \in M(X)$. Then $f(x) * g(x) = 0$, which implies $(f * g)(x) = 0(x)$. Hence $f * g = 0$.

(v) Let $f, g, h \in M(X)$. Then

$$\begin{aligned}((f * g) * h)(x) &= ((f * g)(x)) * h(x) = (f(x) * g(x)) * h(x) \\ &= (f(x) * h(x)) * (g(x) * h(x)) \\ &= ((f * h)(x)) * ((g * h)(x)) \\ &= ((f * h) * (g * h))(x)\end{aligned}$$

for all $x \in X$. This implies $(f * g) * h = (f * g) * (f * h) \in M(X)$. □

THEOREM 3.32. *Let X be a positive implicative AC-algebra and let f_1 and f_2 be two idempotent multipliers on X. If $f_1 \circ f_2 = f_2 \circ f_1$, then $f_1 * f_2$ is an idempotent multiplier on X.*

Proof. We know that $f_1 * f_2$ is a multiplier on X from Theorem 3.31. Now

$$\begin{aligned}((f_1 * f_2) \circ ((f_1 * f_2)(x)) &= (f_1 * f_2)(f_1 * f_2)(x) \\ &= (f_1 * f_2)(f_1(x) * f_2(x)) \\ &= (f_1(f_1(x) * f_2(x)))(f_2(f_1(x) * f_2(x))) \\ &= ((f_1 \circ f_1)(x) * f_2(x)) * ((f_2 \circ f_1)(x) * f_2(x)) \\ &= (f_1(x) * f_2(x)) * ((f_1 \circ f_2)(x) * f_2(x)) \\ &= (f_1(x) * f_2(x)) * (f_1(f_2(x) * f_2(x))) \\ &= (f_1 * f_2)(x) * f_1(0) \\ &= (f_1 * f_2)(x) * 0 = (f_1 * f_2)(x).\end{aligned}$$

Thus $(f_1 * f_2) \circ (f_1 * f_2) = f_1 * f_2$, which implies $f_1 * f_2$ is idempotent. □

Let f be a multiplier of a AC-algebra X. Define a $Kerf$ by

$$Kerf = \{x \in X \mid f(x) = 0\}$$

for all $x \in X$.

THEOREM 3.33. *If f is a multiplier of X and let f be an endomorphism on X, then f is idempotent, i.e., $f^2(x) = f(x)$ for all $x \in K$.*

Proof. Since f is a multiplier on X, we get

$$f(x) * f^2(x) = f(f(x) * f(x)) = f(1) = 1.$$

Hence $f(x) \leq f^2(x)$. Also since f is an endomorphism on X, we have

$$f^2(x) * f(x) = f(f(x) * x) = f(x) * f(x) = 1,$$

which implies $f^2(x) \leq f(x)$. Therefore $f^2(x) = f(f(x)) = f(x)$. □

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