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SHADOWING PROPERTY ON MULTI-VALUED DYNAMICAL SYSTEMS

Hahng-Yun Chu*, Se-Hyun Ku**, and Jong-Suh Park***

Abstract. In this article, we study various notions on multi-valued dynamical systems. We first investigate important tools to express the systems, and prove that the notion of chain recurrence is equivalent to the notion of nonwandering set on compact metric spaces.

1. Introduction

The study for generalization of the theory of original dynamical systems to the theory of multi-valued dynamical systems interests with advent of control theory. In recent decades, such dynamical systems were investigated in several papers and to know more about the systems, see [1, 2, 3, 5, 6].

In this article, we mainly focus on the concepts of recurrence and shadowing property on multi-valued dynamical systems. Now we first introduce the precise definitions which are used in the statements of our results.

Let X be a compact metric space with a metric d and f be a mapping from the space X into its power set 2^X. A *compact-valued mapping* f on X is a set-valued mapping on X with the property that $f(x)$ is a compact subset of X for all $x \in X$. As a sense of relation, the *transpose* of f is the relation $\{(y, x) \in X \times X|\ (x, y) \in f\}$ and denoted by f^t. Note that $(f^t)^t = f$. If f is a map on X, for a subset S of X, we easily have that $f^t(S) = f^{-1}(S)$ where $f^{-1}(S)$ is the preimage of S under f.

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Correspondence should be addressed to Jong-Suh Park, jspark@cnu.ac.kr.

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For an element x of X, a set-valued mapping f is *upper semicontinuous* at x if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, x_1) < \delta$ implies $f(x_1) \subseteq B(f(x), \varepsilon)$. Here, $B(f(x), \varepsilon)$ is an open ε-ball of the compact set $f(x)$. A set-valued mapping f is *lower semicontinuous* at x if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, x_1) < \delta$ implies $f(x) \subseteq B(f(x_1), \varepsilon)$, where $B(f(x_1), \varepsilon)$ is also an open ε-ball of $f(x_1)$. We define that a set-valued mapping f is *continuous* at x if f is both upper semicontinuous at x and lower semicontinuous at x. Note that a compact-valued continuous mapping has a closed graph on the product space $X \times X$.

2. Shadowing in multi-valued dynamics

Let f be a compact-valued continuous mapping on a compact space X and ε a continuous function from X to $\mathbb{R}^+ = (0, \infty)$. Then we can define the functions $m(\varepsilon, f)$ and $M(\varepsilon, f)$ from X to $\mathbb{R}^+$ given by

$$m(\varepsilon, f)(x) := \min \varepsilon(f(x)) \text{ and } M(\varepsilon, f)(x) := \max \varepsilon(f(x))$$

for all $x \in X$. We denote

$$\mathcal{P}(X) := \{\varepsilon : X \longrightarrow (0, \infty) \mid \varepsilon \text{ is continuous } \}.$$

Note that $m(\varepsilon, f) \in \mathcal{P}(X)$ and $M(\varepsilon, f) \in \mathcal{P}(X)$. See [3].

LEMMA 2.1. *Let f be a compact-valued continuous mapping on a compact metric space X. Then f^n is also a compact-valued continuous mapping on X for all $n \in \mathbb{N}$.*

Proof. Using Corollary 2.1 in [3], it is directly obtained. □

Let $\mathcal{F}(X)$ be the set of all nonempty closed subsets in (X, d). For $(A, B) \in \mathcal{F}(X) \times \mathcal{F}(X)$, we define

$$h(A, B) := \sup\{d(x, B) | x \in A\}$$

and

$$D(A, B) := \max\{h(A, B), h(B, A)\}.$$

The number $h(A, B)$ is called the *Hausdorff semidistance* of A from B and the number $D(A, B)$ is called the *Hausdorff distance* between A and B in $\mathcal{F}(X)$. The following two lemmas play so important role to prove our main theorems.

LEMMA 2.2. *Let f be a compact-valued continuous mapping on a compact metric space X. For every $\varepsilon \in \mathcal{P}(X)$, there is an element δ in $\mathcal{P}(X)$ such that $d(x,y) < \delta(x)$ implies $\frac{2}{3}\varepsilon(x) < \varepsilon(y) < 2\varepsilon(x)$ and $D(f(x), f(y)) < m(\varepsilon, f)(x)$.*

Proof. Combining Lemma 2.6 and Lemma 2.9 in [2], we easily prove it. □

LEMMA 2.3. *Let f and f^t be compact-valued continuous mappings on a compact metric space X. Then for every $\varepsilon \in \mathcal{P}(X)$, there exists $\eta \in \mathcal{P}(X)$ such that $M(\eta, f)(x) \leq m(\varepsilon, f)(x)$ for all $x \in X$.*

Proof. See Lemma 2.10 in [2]. □

From now on, we assume that f is a compact-valued continuous mapping on a compact space X.

Let $\varepsilon \in \mathcal{P}(X)$. $\Psi = (x_0, \cdots, x_n)$ is an *ε-chain* for f provided that its length n is at least 1 and that $d(x_i, f(x_{i-1})) < m(\varepsilon, f)(x_{i-1})$ for every i with $1 \leq i \leq n$. A point x is *chain recurrent* provided that for every $\varepsilon \in \mathcal{P}(X)$, there exists an ε-chain from x to x. We denote that $CR(f)$ is the set of all chain recurrent points. A point x is *nonwandering* for f if for every neighborhood U of x and positive integer T, there is a positive integer n with $n > T$ such that $f^n(U) \cap U \neq \emptyset$. We denote the set of all nonwandering points for f in X by $\Omega(f)$. A point x is *periodic* for f with period n if $x \in f^n(x)$ and $x \in f^m(x)$ implies $m \geq n$. We denote the set of all periodic points in X by $Per(f)$.

REMARK 2.4. For a set-valued mapping f from a topological space, it is clear that $\overline{Per(f)} \subseteq \Omega(f)$.

For $\delta \in \mathcal{P}(X)$, a sequence $\{x_i\}_{i \in Z}$ in X is called a *δ-pseudo orbit* if

$$d(f(x_i), x_{i+1}) < m(\delta, f)(x_i)$$

for all i in $\mathbb{Z}$. For an $\varepsilon \in \mathcal{P}(X)$, $\{x_n : n \in \mathbb{Z}\}$ is said to be *ε-traced* by x if $d(f^n(x), x_n) < m(\varepsilon, f^n)(x)$ for all $n \in \mathbb{Z}$. A compact-valued continuous mapping f has *shadowing property* if for every $\varepsilon \in \mathcal{P}(X)$, there exists $\delta \in \mathcal{P}(X)$ such that any δ-pseudo orbit for f can be ε-traced by some point of X.

THEOREM 2.5. *Let f be a compact-valued continuous mapping on a compact metric space. If f satisfies shadowing property, then $CR(f) \subseteq \Omega(f)$.*

Proof. Let $x \in CR(f)$. Let U be a neighborhood of x and T be a positive integer. We can choose a positive real number ε such that $B(x, \varepsilon) \subseteq U$. So $\varepsilon \in \mathcal{P}(\mathcal{X})$ as a constant mapping. Since f has shadowing property, there exists δ in $\mathcal{P}(\mathcal{X})$ such that every δ-pseudo orbit for f can be ε-traced by some point of X. Since $x \in CR(f)$, there is a δ-chain $\{x = x_0,\ x_1, \cdots,\ x_m = x\}$ for f. For $j \in \mathbb{Z}$, we define

$$x_j{}' = x_i$$

where $j = i$ (mod m), $i = 0, 1, 2, ..., m-1$. Then $\{x'_j\}_{i \in Z}$ is a δ-pseudo orbit for f. Since f has shadowing property, there is an element z of X such that

$$d(f^j(z), x_j{}') < m(\delta, f^j)(z) = \delta$$

for all $j \in \mathbb{Z}$. Note that $z \in B(x, \delta)$. Thus we get that for every positive integer n,

$$d(f^{mn}(z), x) = d(f^{mn}(z), x_{mn}{}') < \delta.$$

Then we obtain that

$$\emptyset \neq f^{mn}(z) \cap B(x, \delta) \subseteq f^{mn}(B(x, \delta)) \cap B(x, \delta).$$

Hence we can choose a positive integer n_0 with $mn_0 > T$ such that $f^{mn_0}(U) \cap U \neq \emptyset$. Then we have $x \in \Omega(f)$ which completes this proof. □

THEOREM 2.6. *Let f and f^t be compact-valued continuous mappings on a compact metric space. Then $\Omega(f) \subseteq CR(f)$.*

Proof. Let $x \in \Omega(f)$ and $\varepsilon \in \mathcal{P}(X)$. By Lemma 2.3, there is an element η of $\mathcal{P}(X)$ such that

$$M(\eta, f)(x) \leq \frac{2}{3} m(\varepsilon, f)(x)$$

for all $x \in X$. Using Lemma 2.2, we can pick an element δ of $\mathcal{P}(X)$ such that $d(x, y) < \delta(x)$ implies

$$D(f(x), f(y)) < m(\varepsilon, f)(x)$$

and

$$\frac{2}{3}\eta(x) < \eta(y).$$

Without loss of generality, we can assume that $\delta < \eta$. From the fact that $x \in \Omega(f)$, there is a positive integer n such that

$$f^n(B(x, \delta(x))) \cap B(x, \delta(x)) \neq \emptyset.$$

So we can pick an element z of $f^n(B(x, \delta(x))) \cap (B(x, \delta(x))$.

Now we construct an ε-chain from x to itself. From the choice of z, we can pick up elements $x_0, x_1, \cdots, x_{n-1}$ of X such that

(1) $x_0 \in B(x, \delta(x))$,
(2) $x_i \in f(x_{i-1})$ for $i = 1, 2, 3, \cdots, n-1$,
(3) $z \in f(x_{n-1})$.

Since $d(x, x_0) < \delta(x)$, we get that

$$d(f(x), x_1) < D(f(x), f(x_0)) < m(\varepsilon, f)(x).$$

And it is also clear that $d(f(x_{i-1}), x_i) = 0 < m(\varepsilon, f)(x_{i-1})$. From (3), $d(f(x_{n-1}), x) \leq d(z, x)$. Thus we have that

$$\begin{aligned} d(f(x_{n-1}), x) &\leq d(z, x) < \delta(x) \\ &< \eta(x) < \frac{3}{2}\eta(z) \\ &< \frac{3}{2}M(\eta, f)(x_{n-1}) \\ &< m(\varepsilon, f)(x_{n-1}). \end{aligned}$$

So we conclude that $\{x, x_1, ...x_{n-1}, x\}$ is an ε-chain from x to x. Hence $x \in CR(f)$ which completes the proof. $\square$

Remark 2.7. Let f and f^t be compact-valued continuous mappings on a compact metric space X. Then $\overline{Per(f)} \subseteq \Omega(f) \subseteq CR(f)$.

Using Theorem 2.5 and Theorem 2.6, we directly obtain the following Corollary.

Corollary 2.8. *Let f and f^t be compact-valued continuous mappings on a compact metric space. If f has shadowing property, then $\Omega(f) = CR(f)$.*

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: hychu@cnu.ac.kr

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Department of Mathematics
Chungnam University
Daejeon 305-764, Republic of Korea
E-mail: shku@cnu.ac.kr

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Department of Mathematics
Chungnam University
Daejeon 305-764, Republic of Korea
E-mail: jpark@cnu.ac.kr