

**MULTIPLE L_p ANALYTIC GENERALIZED
FOURIER–FEYNMAN TRANSFORM
ON A FRESNEL TYPE CLASS**

SEUNG JUN CHANG* AND IL YONG LEE**

ABSTRACT. In this paper, we define a class of functional defined on a very general function space $C_{a,b}[0, T]$ like a Fresnel class of an abstract Wiener space. We then define the multiple L_p analytic generalized Fourier–Feynman transform and the generalized convolution product of functionals on function space $C_{a,b}[0, T]$. Finally, we establish some relationships between the multiple L_p analytic generalized Fourier–Feynman transform and the generalized convolution product for functionals in $\mathcal{F}(C_{a,b}[0, T])$.

1. Introduction

Let $C_0[0, T]$ denote one-parameter Wiener space, that is the space of \mathbb{R} -valued continuous functions $x(t)$ on $[0, T]$ with $x(0) = 0$. The concept of an L_1 analytic Fourier–Feynman transform(FFT) for functionals on the Wiener space was introduced by Brue in [2]. In [3], Cameron and Storvick introduced an L_2 FFT on the Wiener space. In [10], Johnson and Skoug developed an L_p FFT theory for $1 \leq p \leq 2$ that extended the results in [2,3] and established various relationships between the L_1 and L_2 theories. In [11], Huffman, Park and Skoug developed an L_p FFT theory on certain classes of functionals defined on Wiener space and they defined a convolution product for functionals in the classes, and they obtained various results for the FFT and convolution product [11,13,12]. On the other hand, in [1], Ahn investigated the L_1 FFT theory on the Fresnel class $\mathcal{F}(B)$ of an abstract

The present research was conducted by the research fund of Dankook University in 2004.

Received by the editors on February 12, 2005.

2000 *Mathematics Subject Classifications*: Primary 60J65, 28C20.

Key words and phrases: generalized Brownian motion process, generalized analytic Feynman integral, generalized analytic Fourier–Feynman transform, generalized convolution product, multiple L_p analytic generalized Fourier–Feynman transform.

Wiener space, and in [4] Chang, Song and Yoo studied the FFT and the first variation on an abstract Wiener space and the Fresnel class $\mathcal{F}(B)$.

In [6], Chang and Choi studied the multiple L_p analytic GFFT on the Banach algebra.

In recent paper [7], Chang and Skoug established various results involving generalized Feynman integrals, and the generalized FFTs(GFFT) for functionals defined on a very general function space $C_{a,b}[0, T]$ rather than on the Wiener space $C_0[0, T]$. The function space was introduced by Chang and Chung in [5].

In [8], Chang and Lee studied the GFFT and CGFFT on a Fresnel type class $\mathcal{F}(C_{a,b}[0, T])$.

In Section 2 of this paper, we introduce the basic concepts and the notations for our research. In Section 3, we study the L_p analytic GFFT and the GCP on a function space $C_{a,b}[0, T]$. Finally, we establish some relationships between the multiple L_p analytic GFFT and the GCP for functionals in $\mathcal{F}(C_{a,b}[0, T])$.

2. Definitions and preliminaries

Let $D = [0, T]$ and let (Ω, \mathcal{B}, P) be a probability measure space. A real-valued stochastic process Y on (Ω, \mathcal{B}, P) and D is called a *generalized Brownian motion process* if $Y(0, \omega) = 0$ almost everywhere and for $0 = t_0 < t_1 < \dots < t_n \leq T$, the n -dimensional random vector $(Y(t_1, \omega), \dots, Y(t_n, \omega))$ is normally distributed with the density function

$$(2.1) \quad K(\vec{t}, \vec{\eta}) = ((2\pi)^n \prod_{j=1}^n (b(t_j) - b(t_{j-1})))^{-1/2} \\ \cdot \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})} \right\}$$

where $\vec{\eta} = (\eta_1, \dots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \dots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$.

As explained in [15,p.18-20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where \mathbb{R}^D is the space of all real valued functions $x(t)$, $t \in D$, and \mathcal{B}^D is the smallest σ -algebra of subsets of \mathbb{R}^D with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on \mathbb{R}^D are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [15,p.187], the probability measure μ induced by Y , taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous functions x on $[0, T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$ is the function space induced by Y , where $\mathcal{B}(C_{a,b}[0, T])$ is the Borel σ -algebra of $C_{a,b}[0, T]$.

A subset B of $C_{a,b}[0, T]$ is said to be scale-invariant measurable(s.i.m.) [9] provided ρB is $\mathcal{B}(C_{a,b}[0, T])$ -measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.). If two functionals F and G defined on $C_{a,b}[0, T]$ are equal s-a.e., then we write $F \approx G$.

Let $L_{a,b}^2[0, T]$ be the set of functions on $[0, T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0, T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$(2.2) \quad L_{a,b}^2[0, T] = \left\{ v : \int_0^T v^2(s) db(s) < \infty \text{ and } \int_0^T v^2(s) d|a|(s) < \infty \right\}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0, t]$.

For $u, v \in L_{a,b}^2[0, T]$, let

$$(2.3) \quad (u, v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)].$$

Then $(\cdot, \cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0, T]$ and $\|u\|_{a,b} = \sqrt{(u, u)_{a,b}}$ is a norm on $L^2_{a,b}[0, T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0, T]$. Furthermore, $(L^2_{a,b}[0, T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthonormal set of real-valued functions of bounded variation on $[0, T]$ such that

$$(\phi_j, \phi_k)_{a,b} = \begin{cases} 0 & , j \neq k \\ 1 & , j = k \end{cases},$$

and for each $v \in L^2_{a,b}[0, T]$, let

$$(2.4) \quad v_n(t) = \sum_{j=1}^n (v, \phi_j)_{a,b} \phi_j(t)$$

for $n = 1, 2, \dots$. Then for each $v \in L^2_{a,b}[0, T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v, x \rangle$ is defined by the formula

$$(2.5) \quad \langle v, x \rangle = \lim_{n \rightarrow \infty} \int_0^T v_n(t) dx(t)$$

for all $x \in C_{a,b}[0, T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0, T]$, the PWZ stochastic integral $\langle v, x \rangle$ exists for μ -a.e. $x \in C_{a,b}[0, T]$.

We denote the function space integral of a $\mathcal{B}(C_{a,b}[0, T])$ -measurable functional F by

$$E[F] = \int_{C_{a,b}[0, T]} F(x) d\mu(x)$$

whenever the integral exists.

Throughout this paper, we will assume that each functional F we consider satisfies the conditions:

$$(2.6) \quad F : C_{a,b}[0, T] \rightarrow \mathbb{C} \text{ is s.i.m. and s-a.e. defined,}$$

and

$$(2.7) \quad E_x[|F(\rho x)|] < \infty \text{ for each } \rho > 0.$$

Next, we state the definition of the generalized analytic Feynman integral.

DEFINITION 2.1. Let \mathbb{C} denote the complex numbers, let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ and let $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \operatorname{Re}\lambda \geq 0\}$. Let F satisfy conditions (2.6) and (2.7) above. If there exists a function $J^*(\lambda)$ analytic in \mathbb{C}_+ such that $J^*(\lambda) = E_x[F(\lambda^{-1/2}x)]$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0, T]$ with parameter λ , and for $\lambda \in \mathbb{C}_+$ we write

$$(2.8) \quad E^{\operatorname{an}\lambda}[F] \equiv E_x^{\operatorname{an}\lambda}[F(x)] = J^*(\lambda).$$

Let $q \neq 0$ be a real number and let F be a functional such that $E^{\operatorname{an}\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$(2.9) \quad E^{\operatorname{anf}_q}[F] \equiv E_x^{\operatorname{anf}_q}[F(x)] = \lim_{\lambda \rightarrow -iq} E^{\operatorname{an}\lambda}[F]$$

where $\lambda \rightarrow -iq$ through values in \mathbb{C}_+ .

Next, we state the definition of the GFFT.

DEFINITION 2.2. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0, T]$, let

$$(2.10) \quad T_\lambda(F)(y) = E_x^{\operatorname{an}\lambda}[F(y+x)].$$

For $p \in (1, 2]$, we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F , by the formula ($\lambda \in \mathbb{C}_+$)

$$(2.11) \quad T_q^{(p)}(F)(y) = \operatorname{l.i.m.}_{\lambda \rightarrow -iq} T_\lambda(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\lambda \rightarrow -iq} \int_{C_{a,b}[0, T]} |T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y)|^{p'} d\mu(y) = 0$$

where $1/p + 1/p' = 1$. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F , by the formula ($\lambda \in \mathbb{C}_+$)

$$(2.12) \quad T_q^{(1)}(F)(y) = \lim_{\lambda \rightarrow -iq} T_\lambda(F)(y)$$

if it exists.

We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$.

DEFINITION 2.3. Let F and G be measurable functionals on $C_{a,b}[0, T]$. For $\lambda \in \tilde{\mathbb{C}}_+$, we define their GCP $(F * G)_\lambda$ (if it exists) by

$$(2.13) \quad (F * G)_\lambda(y) = \begin{cases} E_x^{\text{an}\lambda} [F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})] & , \lambda \in \mathbb{C}_+ \\ E_x^{\text{anf}_q} [F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})] & , \lambda = -iq, q \in \mathbb{R} - \{0\} \end{cases} .$$

REMARK 2.1. (i) When $\lambda = -iq$, we denote $(F * G)_\lambda$ by $(F * G)_q$.

(ii) For any real $q \neq 0$, we briefly describe F_q^* and *F_q of a functional F on $C_{a,b}[0, T]$ as follow:

$$(2.14) \quad F_q^* = (F * 1)_q \quad \text{and} \quad {}^*F_q = (1 * F)_q .$$

The following generalized analytic Feynman integral formula is used several times in this paper.

$$(2.15) \quad E_x[\exp\{i\lambda^{-1/2}\langle v, x \rangle\}] = \exp\left\{-\frac{1}{2\lambda}(v^2, b') + i\lambda^{-1/2}(v, a')\right\}$$

for all $\lambda \in \tilde{\mathbb{C}}_+$ and $v \in L_{a,b}^2[0, T]$ where

$$(2.16) \quad (v, a') = \int_0^T v(t)a'(t)dt = \int_0^T v(t)da(t)$$

and

$$(2.17) \quad (v^2, b') = \int_0^T v^2(t)b'(t)dt = \int_0^T v^2(t)db(t) .$$

In this paper, for each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{-1/2}$ (or $\lambda^{1/2}$) is always chosen to have positive real part.

Let

$$(2.18) \quad C'_{a,b}[0, T] = \left\{ w \in C_{a,b}[0, T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L_{a,b}^2[0, T] \right\} .$$

For $w \in C'_{a,b}[0, T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0, T]$, let $D_t : C'_{a,b}[0, T] \rightarrow L_{a,b}^2[0, T]$ be defined by the formula

$$(2.19) \quad D_t w = z(t) = \frac{w'(t)}{b'(t)} .$$

Then $C'_{a,b} \equiv C'_{a,b}[0, T]$ with inner product

$$(2.20) \quad (w_1, w_2)_{C'_{a,b}} = \int_0^T D_t w_1 D_t w_2 db(t)$$

is a separable Hilbert space. Furthermore, $(C'_{a,b}[0, T], C_{a,b}[0, T], \mu)$ is an abstract Wiener space. For more details, see [14].

Note that for all $w, w_1, w_2 \in C'_{a,b}[0, T]$,

$$(2.21) \quad ((D_t w)^2, b') = \int_0^T (D_t w)^2 db(t) = \|w\|_{C'_{a,b}}^2,$$

$$(2.22) \quad (D_t w, a') = \int_0^T D_t w da(t) = \int_0^T D_t w D_t a db(t) = (w, a)_{C'_{a,b}}$$

and

$$(2.23) \quad \langle D_t w_1, w_2 \rangle = \int_0^T D_t w_1 dw_2(t) = \int_0^T D_t w_1 D_t w_2 db(t) = (w_1, w_2)_{C'_{a,b}}.$$

Next, we define a class of functionals on $C_{a,b}[0, T]$ like a Fresnel class of an abstract Wiener space. Note that the linear operator given by equation (2.19) is an isomorphism. In fact, the inverse operator $D_t^{-1} : L^2_{a,b}[0, T] \rightarrow C'_{a,b}[0, T]$ is given by the formula

$$(2.24) \quad D_t^{-1} z = \int_0^t z(s) db(s)$$

and D_t^{-1} is a bounded operator since

$$(2.25) \quad \begin{aligned} \|D_t^{-1} z\|_{C'_{a,b}} &= \left\| \int_0^t z(s) db(s) \right\|_{C'_{a,b}} = \left(\int_0^T z^2(t) db(t) \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^T z^2(t) d[b(t) + |a|(t)] \right)^{\frac{1}{2}} = \|z\|_{a,b}. \end{aligned}$$

Thus by open mapping theorem, D_t is also bounded and there exist positive real numbers α and β such that $\alpha \|w\|_{C'_{a,b}} \leq \|D_t w\|_{a,b} \leq \beta \|w\|_{C'_{a,b}}$ for all

$w \in C'_{a,b}[0, T]$. Hence we see that the Borel σ -algebra on $(C'_{a,b}[0, T], \|\cdot\|_{C'_{a,b}})$ is given by

$$\mathcal{B}(C'_{a,b}[0, T]) = \{D_t^{-1}(E) : E \in \mathcal{B}(L^2_{a,b}[0, T])\}.$$

Let σ be a complex Borel measure on $\mathcal{B}(L^2_{a,b}[0, T])$. Define a set function f on $\mathcal{B}(C'_{a,b}[0, T])$ by $f(B) = \sigma(D_t(B))$ for each $B \in \mathcal{B}(C'_{a,b}[0, T])$. Then f is a complex Borel measure on $\mathcal{B}(C'_{a,b}[0, T])$. Conversely, let f be a complex Borel measure on $\mathcal{B}(C'_{a,b}[0, T])$. Then the set function σ defined on $\mathcal{B}(L^2_{a,b}[0, T])$ by $\sigma(E) = f(D_t^{-1}(E))$ for each $E \in \mathcal{B}(L^2_{a,b}[0, T])$ is a complex Borel measure on $\mathcal{B}(L^2_{a,b}[0, T])$.

DEFINITION 2.4. *Let*

$$\mathcal{M}(C'_{a,b}[0, T]) = \{f : f = \sigma \circ D_t, \sigma \text{ is a complex Borel measure on } L^2_{a,b}[0, T]\}.$$

The Banach algebra $\mathcal{F}(C_{a,b}[0, T])$ consists of those functionals F on $C_{a,b}[0, T]$ expressible in the form

$$(2.26) \quad F(x) = \int_{C'_{a,b}[0, T]} \exp\{i\langle D_t w, x \rangle\} df(w)$$

for s-a.e. $x \in C_{a,b}[0, T]$ where the associated measure f is an element $\mathcal{M}(C'_{a,b}[0, T])$. We call $\mathcal{F}(C_{a,b}[0, T])$ the Fresnel type class of the function space $C_{a,b}[0, T]$.

REMARK 2.2. (i) $\mathcal{M}(C'_{a,b}[0, T])$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

(ii) One can show that the correspondence $f \rightarrow F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{F}(C_{a,b}[0, T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{C'_{a,b}[0, T]} |df(w)|.$$

3. Transforms and convolutions

In this section, we obtain several results for the GCP of functionals in the class $\mathcal{F}(C_{a,b}[0, T])$.

REMARK 3.1. Let F be an element of $\mathcal{F}(C_{a,b}[0, T])$ whose associated measure f satisfies the condition

$$(3.1) \quad \int_{C'_{a,b}[0, T]} \exp \{ |2q_0|^{-1/2} \|w\|_{C'_{a,b}} \|a\|_{C'_{a,b}} \} |df(w)| < \infty$$

for some $q_0 \in \mathbb{R} - \{0\}$. Then for all $q \in \mathbb{R}$ with $|q| \geq |q_0|$, the generalized analytic Feynman integral $E^{\text{anf}_q}[F]$ of F exists and is given by the formula

$$(3.2) \quad E^{\text{anf}_q}[F] = \int_{C'_{a,b}[0, T]} \exp \left\{ -\frac{i}{2q} \|w\|_{C'_{a,b}}^2 + i \left(\frac{i}{q} \right)^{\frac{1}{2}} (w, a)_{C'_{a,b}} \right\} df(w).$$

In next theorem, we obtain the L_p analytic GFFT $T_q^{(p)}(F)$ of a functional F in $\mathcal{F}(C_{a,b}[0, T])$.

THEOREM 3.1. Let q_0 be a nonzero real number and let F be an element of $\mathcal{F}(C_{a,b}[0, T])$ whose associated measure f satisfies the condition (3.1) above. Then for all $p \in [1, 2]$ and all real q with $|q| \geq |q_0|$, the L_p analytic GFFT of F , $T_q^{(p)}(F)$ exists and is given by the formula

$$(3.3) \quad T_q^{(p)}(F)(y) = \int_{C'_{a,b}[0, T]} \exp \left\{ i \langle D_t w, y \rangle - \frac{i}{2q} \|w\|_{C'_{a,b}}^2 + i \left(\frac{i}{q} \right)^{\frac{1}{2}} (w, a)_{C'_{a,b}} \right\} df(w)$$

for s-a.e. $y \in C_{a,b}[0, T]$. Furthermore, $T_q^{(p)}(F)$ is an element of $\mathcal{F}(C_{a,b}[0, T])$ with associated measure ϕ defined by

$$(3.4) \quad \phi(B) = \int_B \exp \left\{ -\frac{i}{2q} \|w\|_{C'_{a,b}}^2 + i \left(\frac{i}{q} \right)^{\frac{1}{2}} (w, a)_{C'_{a,b}} \right\} df(w)$$

for $B \in \mathcal{B}(C'_{a,b}[0, T])$.

Proof. For $\lambda > 0$ and for s-a.e. $y \in C_{a,b}[0, T]$, using equation (2.10), the Fubini theorem and equation (2.15), we obtain

$$(3.5) \quad \begin{aligned} T_\lambda(F)(y) &= E_x[F(y + \lambda^{-1/2}x)] \\ &= \int_{C'_{a,b}[0, T]} E_x[\exp\{i \langle D_t w, y \rangle + i\lambda^{-1/2} \langle D_t w, x \rangle\}] df(w) \\ &= \int_{C'_{a,b}[0, T]} \exp \left\{ i \langle D_t w, y \rangle - \frac{1}{2\lambda} ((D_t w)^2, b') + \frac{i}{\sqrt{\lambda}} (D_t w, a') \right\} df(w) \end{aligned}$$

$$= \int_{C'_{a,b}[0,T]} \exp \left\{ i \langle D_t w, y \rangle - \frac{1}{2\lambda} \|w\|_{C'_{a,b}}^2 + \frac{i}{\sqrt{\lambda}} (w, a)_{C'_{a,b}} \right\} df(w).$$

But the last expression above is analytic through \mathbb{C}_+ and is continuous on $\tilde{\mathbb{C}}_+$. Also, it is bounded on the region $\Gamma = \{\lambda \in \tilde{\mathbb{C}}_+ : |\operatorname{Im}(\lambda^{-1/2})| \leq |2q_0|^{-1/2}\}$. Thus equation (3.3) is established. Let ϕ be a set function on $\mathcal{B}(C'_{a,b}[0, T])$ defined by equation (3.4). By using condition (3.1) we see that

$$(3.6) \quad \|\phi\| \leq \int_{C'_{a,b}[0,T]} \exp \{ |2q_0|^{-1/2} \|w\|_{C'_{a,b}} \|a\|_{C'_{a,b}} \} |df(w)| < +\infty.$$

Hence we have the desired result. \square

In our next theorem, we obtain the GCP of functionals in $\mathcal{F}(C_{a,b}[0, T])$

THEOREM 3.2. *Let q_0 be the nonzero real number and let F and G be elements of $\mathcal{F}(C_{a,b}[0, T])$ whose associated measures f and g satisfy the condition*

$$(3.7) \quad \int_{C'_{a,b}[0,T]} \exp \{ |4q_0|^{-1/2} \|w\|_{C'_{a,b}} \|a\|_{C'_{a,b}} \} [|df(w)| + |dg(w)|] < \infty.$$

*Then their GCP $(F * G)_q$ exists for all $p \in [1, 2]$ and all real q with $|q| \geq |q_0|$ and is given by the formula*

$$(3.8) \quad \begin{aligned} & (F * G)_q(y) \\ &= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle D_t w_1 + D_t w_2, y \rangle \right. \\ & \quad \left. - \frac{i}{4q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 - 2(w_1, w_2)_{C'_{a,b}}) \right. \\ & \quad \left. + i \left(\frac{i}{2q} \right)^{\frac{1}{2}} ((w_1, a)_{C'_{a,b}} - (w_2, a)_{C'_{a,b}}) \right\} df(w_1) dg(w_2). \end{aligned}$$

Proof. By using (2.13), the Fubini theorem, and (2.15), we have that for $\lambda > 0$,

$$\begin{aligned}
& (F * G)_\lambda(y) \\
&= E_x \left[F \left(\frac{y + \lambda^{-1/2}x}{\sqrt{2}} \right) G \left(\frac{y - \lambda^{-1/2}x}{\sqrt{2}} \right) \right] \\
(3.9) \quad &= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle D_t w_1 + D_t w_2, y \rangle \right. \\
&\quad \left. - \frac{i}{4\lambda} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 - 2(w_1, w_2)_{C'_{a,b}}) \right. \\
&\quad \left. + i \left(\frac{1}{2\lambda} \right)^{\frac{1}{2}} ((w_1, a)_{C'_{a,b}} - (w_2, a)_{C'_{a,b}}) \right\} df(w_1) dg(w_2)
\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0, T]$. But the last expression above is analytic throughout \mathbb{C}_+ , and is continuous on $\tilde{\mathbb{C}}_+$. Thus we have the equation (3.10) above. Let a set function $h : \mathcal{B}(C'_{a,b}[0, T] \times C'_{a,b}[0, T]) \longrightarrow \mathbb{C}$ be defined by

$$\begin{aligned}
(3.10) \quad h(E) &= \int_E \exp \left\{ - \frac{i}{4q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 + 2(w_1, w_2)_{C'_{a,b}}) \right. \\
&\quad \left. + i \left(\frac{i}{2q} \right)^{\frac{1}{2}} ((w_1, a)_{C'_{a,b}} - (w_2, a)_{C'_{a,b}}) \right\} df(w_1) dg(w_2)
\end{aligned}$$

for each $E \in \mathcal{B}(C'_{a,b}[0, T] \times C'_{a,b}[0, T])$. Then h is a complex Borel measure on $\mathcal{B}(C'_{a,b}[0, T] \times C'_{a,b}[0, T])$. Now we define a function $\varphi : C'_{a,b}[0, T] \times C'_{a,b}[0, T] \longrightarrow C'_{a,b}[0, T]$ by

$$(3.11) \quad \varphi(w_1, w_2) = \frac{1}{\sqrt{2}}(w_1 + w_2).$$

Then φ is continuous and so it is Borel measurable. Let $\tilde{h} = h \circ \varphi^{-1}$. By the condition (3.7) above, we that for real q with $|q| \geq |q_0|$

$$\begin{aligned}
\|\tilde{h}\| &= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} |dh(w_1, w_2)| \\
&\leq \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \left| \exp \left\{ - \frac{i}{4q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 - 2(w_1, w_2)_{C'_{a,b}}) \right. \right.
\end{aligned}$$

$$\begin{aligned}
(3.12) \quad & + i \left(\frac{i}{2q} \right)^{\frac{1}{2}} \left((w_1, a)_{C'_{a,b}} - (w_2, a)_{C'_{a,b}} \right) \left| df(w_1) \right| |dg(w_2)| \\
& \leq \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{1}{\sqrt{4q_0}} \int_0^T |w_1(s)| |da(s)| \right\} |df(w_1)| \\
& \quad \cdot \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{1}{\sqrt{4q_0}} \int_0^T |w_2(s)| |da(s)| \right\} |dg(w_2)| < \infty.
\end{aligned}$$

Hence $\tilde{h} = h \circ \varphi^{-1}$ belongs to $\mathcal{M}(C'_{a,b}[0, T])$ and

$$(3.13) \quad (F * G)_q(y) = \int_{C'_{a,b}[0,T]} \exp\{i\langle r, y \rangle\} d h(r)$$

for s-a.e. $y \in C'_{a,b}[0, T]$. Hence $(F * G)_q$ exists and is given by (3.8) for all real q with $|q| \geq |q_0|$ and it belong to $\mathcal{F}(C_{a,b}[0, T])$. \square

REMARK 3.2. Let F , f , and q_0 be as in Theorem 3.2. Then for all real q with $|q| \geq |q_0|$, $*F_q$ and F_q^* are in $\mathcal{F}(C_{a,b}[0, T])$.

THEOREM 3.3. Let F , G , f , g , and q_0 be as in Theorem 3.2. Then for all $p \in [1, 2]$ and all real q with $|q| \geq |q_0|$,

$$(3.14) \quad T_q^{(p)}((F * G)_q)(y) = \left(T_q^{(p)}(F_q^*)(y) \right) \left(T_q^{(p)}(*G_q)(y) \right)$$

for s-a.e. $y \in C_{a,b}[0, T]$, where F_q^* and $*G_q$ are given by (2.13). Also, both of the expressions in (3.14) are given by the expression

$$\begin{aligned}
(3.15) \quad & \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{1}{\sqrt{2}} \langle D_t w_1 + D_t w_2, y \rangle - \frac{i}{2q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2) \right. \\
& \quad \left. + 2i \left(\frac{i}{2q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} \right\} df(w_1) dg(w_2).
\end{aligned}$$

Proof. By using (2.10), (2.13), the Fubini theorem, and (2.15), we have that for $\lambda > 0$,

$$(3.16) \quad T_\lambda((F * G)_\lambda)(y) = T_\lambda(F_\lambda^*)(y) T_\lambda(G_\lambda^*)(y)$$

for s-a.e. $y \in C_{a,b}[0, T]$. But both of the expressions on the right-hand side of equation (3.16) are analytic functions of λ throughout \mathbb{C}_+ , and are continuous functions of λ on $\tilde{\mathbb{C}}_+$ for all $y \in C_{a,b}[0, T]$. By using (3.7), $T_q^{(p)}((F * G)_q)$ exists for all real q with $|q| \geq |q_0|$ and is given by (3.14) for all desired values of p and q . \square

THEOREM 3.4. *Let $F, G, f, g,$ and q_0 be as in Theorem 3.3. Then*

$$\begin{aligned}
 (3.17) \quad & \int_{C_{a,b}[0,T]}^{anf-q} T_q^{(p)}((F * G)_q)(y) d\mu(y) \\
 &= \int_{C_{a,b}[0,T]}^{anf-q} T_q^{(p)}(F_q^*)(y) T_q^{(p)}(*G_q)(y) d\mu(y) \\
 &= \int_{C_{a,b}[0,T]}^{anf_q} (F_{-q}^*)_{\frac{q}{2}}^*(\sqrt{2}y) (G_{-q}^*)_{\frac{q}{2}}^*(-\sqrt{2}y) d\mu(y).
 \end{aligned}$$

Proof. Fix p and q . Then for $\lambda > 0$, using (3.9), the Fubini theorem and (3.3), we have

$$\begin{aligned}
 & \int_{C_{a,b}[0,T]} T_q^{(p)}((F * G)_q)(y/\sqrt{\lambda}) d\mu(y) \\
 &= \int_{C_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{1}{\sqrt{2\lambda}} \langle D_t w_1 + D_t w_2, y \rangle \right. \\
 & \quad \left. - \frac{i}{2q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2) + 2i \left(\frac{i}{2q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} \right\} df(w_1) dg(w_2) d\mu(y) \\
 &= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ -\frac{1}{4\lambda} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 + 2(w_1, w_2)_{C'_{a,b}}) \right. \\
 & \quad \left. \frac{i}{\sqrt{2\lambda}} ((w_1, a)_{C'_{a,b}} + (w_2, a)_{C'_{a,b}}) - \frac{i}{2q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2) \right. \\
 (3.18) \quad & \left. + 2i \left(\frac{i}{2q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} \right\} df(w_1) dg(w_2).
 \end{aligned}$$

But the last expression is an analytic function of λ throughout $\tilde{\mathbb{C}}_+$ and is

continuous throughout $\tilde{\mathbb{C}}_+$, and so letting $\lambda = -i(-q) = iq$, we obtain that

$$\begin{aligned}
& \int_{C_{a,b}[0,T]}^{anf-q} T_q^{(p)}((F * G)_q)(y) d\mu(y) \\
&= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{i}{4q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 + 2(w_1, w_2)_{C'_{a,b}}) \right. \\
(3.19) \quad & \left. + i \left(\frac{-i}{2q} \right)^{\frac{1}{2}} ((w_1, a)_{C'_{a,b}} + (w_2, a)_{C'_{a,b}}) \right. \\
& \quad \left. - \frac{i}{2q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2) + 2i \left(\frac{i}{2q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} \right\} df(w_1) dg(w_2) \\
&= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ - \frac{i}{4q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 - 2(w_1, w_2)_{C'_{a,b}}) \right. \\
& \quad \left. + i \left(\frac{-i}{2q} \right)^{\frac{1}{2}} ((w_1, a)_{C'_{a,b}} + (w_2, a)_{C'_{a,b}}) + 2i \left(\frac{i}{2q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} \right\} df(w_1) dg(w_2).
\end{aligned}$$

Clearly, the condition (3.7) will imply the existence of (3.19). On the other hand, using (2.14),(3.8), the Fubini theorem, and (3.2), we obtain that

$$\begin{aligned}
(F_{-q}^*)_{\frac{q}{2}}^*(\sqrt{2}y) &= \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle D_t w_1, y \rangle \right. \\
(3.20) \quad & \left. + i \left(\frac{i}{2q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} + i \left(\frac{-i}{2q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} \right\} df(w_1)
\end{aligned}$$

and

$$\begin{aligned}
(G_{-q}^*)_{\frac{q}{2}}^*(-\sqrt{2}y) &= \int_{C'_{a,b}[0,T]} \exp \left\{ - \frac{i}{\sqrt{2}} \langle D_t w_2, y \rangle \right. \\
(3.21) \quad & \left. + i \left(\frac{i}{2q} \right)^{\frac{1}{2}} (w_2, a)_{C'_{a,b}} + i \left(\frac{-i}{2q} \right)^{\frac{1}{2}} (w_2, a)_{C'_{a,b}} \right\} dg(w_2)
\end{aligned}$$

s-a.e. $y \in C_{a,b}[0, T]$. By using (3.20) and (3.21), we have that for $\lambda > 0$

$$\begin{aligned}
& \int_{C_{a,b}[0,T]} (F_{-q}^*)_{\frac{q}{2}}^*(\sqrt{2}y/\sqrt{\lambda})(G_{-q}^*)_{\frac{q}{2}}^*(-\sqrt{2}y/\sqrt{\lambda})d\mu(y) \\
&= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ -\frac{1}{4\lambda}(\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 - 2(w_1, w_2)_{C'_{a,b}}) \right. \\
(3.22) \quad & \left. + \frac{i}{\sqrt{2\lambda}}((w_1, a)_{C'_{a,b}} - (w_2, a)_{C'_{a,b}}) + i\left(\frac{i}{2q}\right)^{\frac{1}{2}}((w_1, a)_{C'_{a,b}} - (w_2, a)_{C'_{a,b}}) \right. \\
& \left. + i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}((w_1, a)_{C'_{a,b}} + (w_2, a)_{C'_{a,b}}) \right\} df(w_1)dg(w_2).
\end{aligned}$$

But the last expression above is an analytic function of λ throughout $\tilde{\mathbb{C}}_+$ and is continuous throughout on $\tilde{\mathbb{C}}_+$ and so letting $\lambda \rightarrow -iq$ we obtain that

$$\begin{aligned}
(3.23) \quad & \int_{C_{a,b}[0,T]}^{anf_q} (F_{-q}^*)_{\frac{q}{2}}^*(\sqrt{2}y)(G_{-q}^*)_{\frac{q}{2}}^*(-\sqrt{2}y)d\mu(y) \\
&= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ -\frac{i}{4q}(\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 - 2(w_1, w_2)_{C'_{a,b}}) \right. \\
& \left. + 2i\left(\frac{i}{2q}\right)^{\frac{1}{2}}(w_1, a)_{C'_{a,b}} + i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}((w_1, a)_{C'_{a,b}} + (w_2, a)_{C'_{a,b}}) \right\} df(w_1)dg(w_2).
\end{aligned}$$

Now (3.19) and (3.23) together yield (3.17). \square

REMARK 3.4. In Theorem 3.4 above, if $a(t) \equiv 0$, then for all $q \neq 0$,

$$(3.24) \quad T_q^{(p)}(F_q^*)(F)(y) = T_q^{(p)}(F)(y/\sqrt{2}) \text{ and } T_q^{(p)}(*G_q)(F)(y) = T_q^{(p)}(G)(y/\sqrt{2})$$

for s-a.e. $y \in C_{a,b}[0, T]$. Furthermore

$$(F_{-q}^*)_{\frac{q}{2}}^*(\sqrt{2}y) = F(y/\sqrt{2}) \quad \text{and} \quad (G_{-q}^*)_{\frac{q}{2}}^*(-\sqrt{2}y) = G(-y/\sqrt{2}).$$

Hence we have the following Parseval's identity

$$\begin{aligned}
& \int_{C_{a,b}[0,T]}^{anf-q} T_q^{(p)}((F * G)_q)(y) d\mu(y) \\
&= \int_{C_{a,b}[0,T]}^{anf-q} T_q^{(p)}(F)(y/\sqrt{2}) T_q^{(p)}(G)(y/\sqrt{2}) d\mu(y) \\
&= \int_{C_{a,b}[0,T]}^{anf-q} F(y/\sqrt{2}) G(-y/\sqrt{2}) d\mu(y).
\end{aligned}$$

4. Multiple L_p analytic GFFT and the GCP

In this section we establish some relationships between the multiple L_p analytic GFFT and the GCP of functionals in $\mathcal{F}(C_{a,b}[0, T])$.

Definition 4.1 Let F be a measurable functional defined on $C_{a,b}[0, T]$ and define a transform $(T_\gamma)^{(n)}$ ($\gamma > 0$) of F by

$$(4.1) \quad (T_\gamma)^{(n)}(F) = \underbrace{(T_\gamma \circ \cdots \circ T_\gamma)}_{n\text{-times}}(F)$$

that is, $(T_\gamma)^{(n)}$ means the n -times composition of (T_γ) , where T_γ is given by (2.10) in Definition 2.2 and n is a nonnegative integer. When γ is in \mathbb{C}_+ , the transform $(T_\gamma)^{(n)}(F)$ means the analytic extension of $(T_\gamma)^n(F)$ ($\gamma > 0$) as the function of $\lambda \in \mathbb{C}_+$. Let $(T_\lambda)^{(n)}(F)$ be analytic extension of $(T_\gamma)^n(F)$ as a function of $\lambda \in \mathbb{C}_+$. In case that $1 < p \leq 2$, for each $q \in \mathbb{R} - \{0\}$, we define the multiple L_p analytic GFFT $(T_q)^{(n)}(F)$ of F by

$$(4.2) \quad (T_q^{(p)})^{(n)}(F) = \text{l.i.m.}_{\lambda \rightarrow -iq} (T_\lambda)^{(n)}(F),$$

where λ approaches $-iq$ through \mathbb{C}_+ . In case that $p = 1$, for each $q \in \mathbb{R} - \{0\}$, we define the multiple L_1 analytic GFFT $(T_q^{(1)})^{(n)}(F)$ of F by

$$(4.3) \quad (T_q^{(1)})^n(F) = \lim_{\lambda \rightarrow -iq} (T_\lambda)^{(n)}(F),$$

where λ approaches $-iq$ through \mathbb{C}_+ .

Note that $(T_\lambda)^{(0)}(F) \equiv F \equiv (T_q^{(p)})^{(0)}$, $(T_\lambda)^{(1)}(F) \equiv T_\lambda(F)$, and $(T_q^{(p)})^{(1)} \equiv T_q^{(p)}(F)$.

We have already shown that for $F \in \mathcal{F}(C_{a,b}[0, T])$ with condition (3.1), the L_p GFFT $T_q^{(p)}(F)$ belongs to the Fresnel type class $\mathcal{F}(C_{a,b}[0, T])$. Hence by using the mathematical induction and proceeding as in the proof of Theorem 3.1, we can obtain the following theorem.

THEOREM 4.1. *Let q_0 be a nonzero real number and let n be a nonnegative integer. Let $F \in \mathcal{F}(C_{a,b}[0, T])$ be given by (3.26) whose associated measure f satisfies the condition*

$$(4.4) \quad \int_{C'_{a,b}[0, T]} \exp \{n|2q_0|^{-1/2} \|w\|_{C'_{a,b}} \|a\|_{C'_{a,b}}\} |df(w)| < \infty.$$

Then for all $p \in [1, 2]$ and all real q with $|q| \geq |q_0|$, the multiple L_p analytic GFFT $(T_q^{(p)})^{(n)}(F)$ exists and is given by

$$(4.5) \quad \begin{aligned} & (T_q^{(p)})^{(n)}(F) \\ &= \int_{C'_{a,b}[0, T]} \exp \left\{ i \langle D_t w, y \rangle - \frac{in}{2q} \|w\|_{C'_{a,b}}^2 + in \left(\frac{i}{q} \right)^{\frac{1}{2}} (w, a)_{C'_{a,b}} \right\} df(w) \end{aligned}$$

for s-a.e. $y \in C_{a,b}[0, T]$. And $(T_q^{(p)})^{(n)}(F)$ is an element of $\mathcal{F}(C_{a,b}[0, T])$ with associated measure

$$\phi_n(E) = \int_E \exp \left\{ -\frac{in}{2q} \|w\|_{C'_{a,b}}^2 + in \left(\frac{i}{q} \right)^{\frac{1}{2}} (w, a)_{C'_{a,b}} \right\} df(w)$$

for $E \in \mathcal{B}(C'_{a,b}[0, T])$. Note that (4.5) is reduced to (3.3), if we take $n = 1$ in (4.5).

Next, we obtain the GCP of the multiple L_p analytic GFFT's of functionals in $\mathcal{F}(C_{a,b}[0, T])$.

THEOREM 4.2. *Let q_0 be a nonzero real number and let n be nonnegative integer. Let F and G be elements of $\mathcal{F}(C_{a,b}[0, T])$ whose associated measures f and g satisfy the condition*

$$(4.6) \quad \int_{C'_{a,b}[0, T]} \exp \{n|2q_0|^{-1/2} \|w\|_{C'_{a,b}} \|a\|_{C'_{a,b}}\} [|df(w)| + |dg(w)|] < \infty.$$

Then for all $p \in [1, 2]$, all q with $|q| \geq |q_0|$ and nonnegative integer m , the GCP $((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q(y)$ exists and is given by (4.7) below. Furthermore $((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q(y)$ is an elements of $\mathcal{F}(C_{a,b}[0, T])$.

Proof. By using (4.5) and (3.8) we observe that for all $p \in [1, 2]$ and all q with $|q| \geq |q_0|$

$$\begin{aligned}
& ((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q(y) \\
&= \int_{C'_{a,b}[0, T]} \int_{C'_{a,b}[0, T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle D_t w_1 + D_t w_2, y \rangle - \frac{in}{2q} \|w_1\|_{C'_{a,b}}^2 - \frac{im}{2q} \|w_2\|_{C'_{a,b}}^2 \right. \\
&\quad \left. + in \left(\frac{i}{q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} + im \left(\frac{i}{q} \right)^{\frac{1}{2}} (w_2, a)_{C'_{a,b}} \right. \\
(4.7) \quad &\quad \left. - \frac{i}{4q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 + 2(w_1, w_2)_{C'_{a,b}}) \right. \\
&\quad \left. + i \left(\frac{i}{2q} \right)^{\frac{1}{2}} ((w_1, a)_{C'_{a,b}} + (w_2, a)_{C'_{a,b}}) \right\} df(w_1) dg(w_2)
\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0, T]$. Furthermore, proceeding as in the proof of Theorem 3.2 above and using (4.6), we see that $((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q(y)$ is an element of $\mathcal{F}(C_{a,b}[0, T])$ \square

Note that (4.7) is reduced to (3.8), if we take $m = n = 0$ in (3.8).

In our next theorem, we obtain the multiple L_p analytic GFFT of the convolution product for two functionals in $\mathcal{F}(C_{a,b}[0, T])$.

THEOREM 4.3. *Let F, G, f, g and q_0 be as in Theorem 4.2. Then for all $p \in [1, 2]$ and all real q the following equation with $|q| \geq |q_0|$,*

$$\begin{aligned}
& (T_q^{(p)})^{(n)}((F * G)_q)(y) \\
&= \int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle D_t w_1 + D_t w_2, y \rangle \right. \\
&\quad \left. - \frac{i}{4q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 - 2(w_1, w_2)_{C'_{a,b}}) \right. \\
&\quad \left. + i \left(\frac{i}{2q} \right)^{\frac{1}{2}} ((w_1, a)_{C'_{a,b}} - (w_2, a)_{C'_{a,b}}) \right. \\
&\quad \left. - \frac{in}{4q} (\|w_1\|_{C'_{a,b}}^2 + \|w_2\|_{C'_{a,b}}^2 + 2(w_1, w_2)_{C'_{a,b}}) \right. \\
&\quad \left. + in \left(\frac{i}{2q} \right)^{\frac{1}{2}} ((w_1, a)_{C'_{a,b}} + (w_2, a)_{C'_{a,b}}) \right\} df(w_1) dg(w_2)
\end{aligned} \tag{4.8}$$

holds of s-a.e. $y \in C_{a,b}[0, T]$, where n is a nonnegative integer. Furthermore, $(T_q^{(p)})^{(n)}((F * G)_q)(y)$ is an element of $\mathcal{F}(C_{a,b}[0, T])$.

Proof. By using equations (3.8) and (4.5), we can easily obtain the equation (4.8) above. Moreover, the condition (4.6) will imply the existence of the equation (4.8). \square

Finally, we show that the L_p analytic GFFT of the GCP of the multiple L_p analytic GFFT's of the transforms for functionals in $\mathcal{F}(C_{a,b}[0, T])$.

THEOREM 4.4. *Let F, G, f, g and q_0 be as in Theorem 4.2. Then for all $p \in [1, 2]$ and all real q the following equation with $|q| \geq |q_0|$,*

$$\begin{aligned}
& T_q^{(p)}(((T_q^{(p)})^{(n)}(F) * (T_q^{(p)})^{(m)}(G))_q)(y) \\
&= (T_{q/2}^{(p)})^{(n)}(T_q^{(p)}(F_q^*))_q(y) (T_{q/2}^{(p)})^{(m)}(T_q^{(p)}(*G_q))_q(y)
\end{aligned} \tag{4.9}$$

holds for s-a.e. $y \in C_{a,b}[0, T]$, where F_q^* and $*G_q$ are as in (2.14). Also, both expressions in (4.9) are given by the expression

$$\int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle D_t w_1 + D_t w_2, y \rangle - \frac{i(n+1)}{2q} \|w_1\|_{C'_{a,b}}^2 \right. \\ \left. - \frac{i(m+1)}{2q} \|w_2\|_{C'_{a,b}}^2 + in \left(\frac{i}{q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} \right. \\ \left. + im \left(\frac{i}{q} \right)^{\frac{1}{2}} (w_2, a)_{C'_{a,b}} + i\sqrt{2} \left(\frac{i}{q} \right)^{\frac{1}{2}} (w_1, a)_{C'_{a,b}} \right\} df(w_1) dg(w_2).$$

Furthermore, the transform $T_q^{(p)}((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q)$ is an element of $\mathcal{F}(C_{a,b}[0, T])$.

Proof. By using (4.7), (3.8) and (3.3), we can obtain the equation (4.9) above. \square

REMARK 4.1. In Theorem 4.4 above, if $a(t) \equiv 0$, then

$$(4.10) \quad (T_{q/2}^{(p)})^{(n)}(T_q^{(p)}(F)(\cdot/\sqrt{2}))(y) = (T_q^{(p)})^{(n+1)}(F)(y/\sqrt{2})$$

and

$$(4.11) \quad (T_{q/2}^{(p)})^{(m)}(T_q^{(p)}(G))(\cdot/\sqrt{2})(y) = (T_q^{(p)})^{(m+1)}(G)(y/\sqrt{2})$$

Hence by using (3.19), (4.10) and (4.11) we obtain that

$$T_q^{(p)}((T_q^{(p)})^{(n)}(F) * ((T_q^{(p)})^{(m)}(G))_q)(y) \\ = (T_q^{(p)})^{(n+1)}(F)(y/\sqrt{2})(T_q^{(p)})^{(m+1)}(G)(y/\sqrt{2})$$

for s-a.e. $y \in C_{a,b}[0, T]$.

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DEPARTMENT OF MATHEMATICS
DANKOOK UNIVERSITY
CHEONAN 330–714, KOREA
E-mail: sejchang@dankook.ac.kr

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DEPARTMENT OF MATHEMATICS
DANKOOK UNIVERSITY
CHEONAN 330–714, KOREA
E-mail: iylee@dankook.ac.kr