PORTFOLIO SELECTION WITH INCOME RISK: A NEW APPROACH

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Abstract. The optimal portfolio choice problem with a stochastic income is considered in continuous-time framework. We provide a novel approach to treat the stochastic income when the market is complete. The developed method is useful to obtain closed-form solutions of the problems under borrowing constraints.

1. Introduction

The portfolio choice problem helps to understand the consumption and investment behavior of individual agents or households. Since the most economic agents earn wages from their labor it is natural to include income stream into their wealth process. In addition, compared to the deterministic income stream, the stochastic income reflects the market condition so it gives more realistic feature.

In continuous time framework, there have been numerous studies about the effect of income risk on consumption and portfolio. In complete market, Merton [11] first considers the problem with stochastic income and recently, more extended versions have been investigated in Farhi and Panageas [5], Munk and Sørensen [13], and Dybvig and Liu [4]. Moreover, He and Pagés [6], El Karoui and Jeanblac-Piqué [9], and Detemple and Serrat [2] resolve the problems in the presence of borrowing constraints. Unfortunately, however, if the income risk is not perfectly hedged by risky asset so the market is incomplete, it is difficult to get an explicit solution. In particular, Karatzas et al.[8], Koo [10] and Munk [12] obtain semi-closed form solutions and investigate the

impact of income risk on the optimal consumption and investment numerically. The uniqueness and existence of the optimal controls without explicit expressions are well described in Duffie et al. [3].

In this paper we provide a novel approach to tackle the problem with stochastic income in complete market. The method is easy to understand and useful to treat the wealth constraint. In particular, we apply the method to the problem with borrowing constraints which is not allowed to borrow against future income. The explicit solutions are obtained by resolving a free-boundary ODE (ordinary differential equation).

This paper is organized as follows. Section 2 explains a market environment and reviews existing methods. Section 3 provides the new approach and explicit solutions. The developed method is applied in the presence of borrowing constraint in Section 4.

2. Environment

In financial market there are two types of financial instruments which are risky assets and riskless assets. The market is complete so suffice it to consider that the financial assets consist of a risky asset and a riskless asset by mutual fund separation theorem. A portfolio of risky assets can be considered as one risky stock and it is supposed to follow a geometric Brownian motion with constant coefficients μ_s and σ_s:

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s dB_t,$$

where B_t is a standard Brownian motion under the regular probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is a sample space and its filtration is given by $\mathcal{F}$ with a probability measure $\mathbb{P}$. The riskless asset can be a bond or bank account which gives an interest $r > 0$ so that it evolves

$$\frac{dS_t^0}{S_t^0} = rdt.$$

The economic agent receives a stochastic income stream from his/her labor. For the market completeness, we presume that the source of uncertainty in labor income is same with the market risk, B_t. Thus the income process I_t is governed by

$$\frac{dI_t}{I_t} = \mu_I dt + \sigma_I dB_t, \qquad I_0 = i,$$

where μ_I and σ_I are positive constant coefficients. Since the income risk is perfectly correlated with a market risk, if the market receives a positive shock, the income grows up when σ_I is positive, and vice versa.

A consumption rate c_t is assumed to be $\mathcal{F}$-adapted with $\int_0^\infty c_t dt < \infty$ a.s. and the investment amount in risky asset denoted by π_t is also supposed to be $\mathcal{F}_t$-progressively measurable and satisfy $\int_0^\infty \pi_t{}^2 dt < \infty$ a.s.

Then the wealth process X_t is governed by

$$(2.1) \quad dX_t = (X_t - \pi_t)\frac{dS_t^0}{S_t^0} + \pi_t \frac{dS_t}{S_t} - c_t dt + I_t dt$$

$$= [rX_t + \pi_t(\mu_s - r) - c_t + I_t]dt + \pi_t \sigma_s dB_t, \qquad X_0 = x.$$

The infinitely-lived agent has a CRRA(constant relative risk aversion) utility with a risk aversion γ, which is represented by

$$u(c_t) = \frac{1}{1-\gamma}c_t^{1-\gamma}, \qquad \gamma > 0.$$

It is straightforward to check the utility function is non-decreasing and concave. When the discount factor is given by a constant $\beta > 0$, he/she faces a maximization problem of the expected utility by choosing consumption rate and investment in risky asset so that the value function is defined by

$$(2.2) \qquad V(x,i) := \max_{c_s,\pi_s} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{1}{1-\gamma}c_t^{1-\gamma}dt\right],$$

subject to the wealth constraint (2.1).

Before introducing a new approach, we briefly review two well-known methods to tackle the problem. They are dynamic programming principle and martingale method. When the market price of risk θ is defined by $\theta := (\mu_s - r)/\sigma_s$, the dynamic programming principle gives a Hamilton-Jacobi-Bellman(HJB) equation as

$$\beta V(x,i) = \max_{\{c_t,\pi_t\}}\left\{\frac{\gamma}{1-\gamma}c_t^{1-\gamma} + (rX_t + I_t - c_t + \sigma_s\theta\pi_t)V_x(x,i)\right.$$

$$\left. + \mu_I I_t V_i(x,i) + \frac{1}{2}\sigma_s^2\pi_t^2 V_{xx}(x,i) + \frac{1}{2}\sigma_I^2 I_t^2 V_{ii}(x,i) + \sigma_I\sigma_s\pi_t V_{xi}(x,i)\right\},$$

where $V_x, V_i, V_{xx}, V_{xi},$ and V_{ii} are partial derivatives. Then the optimal consumption rate can be obtained from first order conditions:

$$c_t^* = (V_x)^{-\frac{1}{\gamma}}, \qquad \pi_t^* = -\frac{\theta V_x}{\sigma_s V_{xx}} - I_t\frac{\sigma_I V_{xi}}{\sigma_s V_{xx}}.$$

Fortunately, even if there is no technical modification this two dimensional PDE can be solved explicitly by conjecturing the value function. However it is nontrivial if there exist constraints related to consumption or wealth, which means that the PDE is not solvable anymore except for the extreme cases.

In case of martingale method, it needs to transform the wealth dynamics (2.1) into a static budget constraint. To derive it, we have to find a unique risk neutral martingale measure first. After the dynamic budget is rewritten as the integral form, taking an expectation under the risk neutral measure gives a static form. However, to treat the static budget, it should be converted into the static form under the original measure. The Bayes' rule can be applied for inverting measure. The following is the final static budget we can derive.

$$\mathbb{E}\left[\int_0^\infty H_t c_t dt\right] \leq x + \mathbb{E}\left[\int_0^\infty H_t I_t dt\right],$$

where H_t is the pricing kernel defined by $H_t = e^{-rt}M_t$. The process M_t is the Radon-Nikodym derivative given by $M_t := \exp\left(-\frac{1}{2}\theta^2 t - \theta B_t\right)$, which is used to define the risk neutral measure. Then the value function is obtained by the duality approach.

In this paper, we provide a novel method to obtain the explicit form solution. It is easy to understand and has a potential for applying to the extended models with retirement choice or under constraints.

3. New approach

Using the basic idea and results from two existing methodologies the new approach can be regarded as a somewhat combination of the two approaches. From the dynamic programming principle, after substituting the optimal consumption rate and portfolio amount to the HJB equation, the value function $V(x, i)$ should satisfy the following PDE:

$$\beta V = \frac{\gamma}{1-\gamma}V_x^{1-\frac{1}{\gamma}} + (rX+I)V_x + \mu_I I V_i - \frac{1}{2}\theta^2\frac{V_x^2}{V_{xx}}$$

(3.1) $$\qquad\qquad - I\sigma_I\theta\frac{V_x V_{xi}}{V_{xx}} - \frac{1}{2}I^2\sigma_I^2\frac{V_{xi}^2}{V_{xx}} + \frac{1}{2}V_{ii}I^2\sigma_I^2.$$

This is a non-linear second order PDE which can be resolved by conjecturing a solution or transforming to the functions with another variables. The latter case is well-established in Cox and Huang [1] and Karatzas et al. [7].

We propose a new transformation as

(3.2) $V(X_t, I_t) = I_t^{1-\gamma} \left\{ \varphi(z_t) - z_t \varphi'(z_t) \right\}$ & $X_t = -I_t \varphi'(z_t).$

with the variable z_t as a new state variable.

ASSUMPTION 3.1. *The function $\varphi(z_t)$ is second order differentiable.*

Note that this transformation can be actually obtained from the traditional martingale approach as in Cox and Huang [1]. However, there need some technical assumptions and complicated procedure to derive. By imposing the value function as the form in (3.2), it is easier to reduce the two dimensional PDE (3.1) into an ODE. The second-order differential assumption of the function $\varphi(\cdot)$ is the only technical assumption we need. For the notational convenience, we omit the subscript t in the sequel.

Let's denote the first and second derivatives of function $\varphi(z)$ by $\varphi'(z)$ and $\varphi''(z)$. Then from the simple calculation we have

$$\frac{\partial z}{\partial I} = -\frac{\varphi'}{I\varphi''}, \qquad \frac{\partial z}{\partial X} = -\frac{1}{I\varphi''}.$$

LEMMA 3.2. *The partial derivatives of the value function $V(x, i)$ in (3.2) are rewritten as follows:*

$$V_x(X, I) = zI^{-\gamma}, \qquad V_i(X, I) = I^{-\gamma} \left((1-\gamma)\varphi + \gamma z \varphi' \right),$$
$$V_{xx}(X, I) = -I^{-1-\gamma}\frac{1}{\varphi''}, \quad V_{xi}(X, I) = -\left(\frac{\varphi'}{\varphi''} + \gamma z \right) I^{-\gamma-1},$$

$$V_{ii}(X, I) = \left(-\frac{\varphi'^2}{\varphi''} - \gamma z \varphi' - \gamma \varphi + \gamma^2 \varphi - \gamma^2 z \varphi' \right) I^{-\gamma-1}.$$

If we substitute the partial derivatives above into the PDE in (3.1), the PDE transformed into the following ODE,

(3.3) $\frac{1}{2}\sigma_z^2 z^2 \varphi''(z) - (r_I - \hat{\beta}) z \varphi'(z) - \hat{\beta}\varphi(z) + \frac{\gamma}{1-\gamma} z^{1-\frac{1}{\gamma}} + z = 0,$

where the constants are given by

$$\sigma_z = \gamma \sigma_I - \theta, \quad \hat{\beta} = \beta - (1-\gamma)\mu_I - \frac{1}{2}\gamma(\gamma-1)\sigma_I^2, \quad r_I = r - \mu_I + \sigma_I \theta.$$

Note that ODE (3.3) is a Cauchy-Euler equation of degree two with no boundary condition. Thus, it has no general solution by growth condition and the particular solution only exists. We summarize our main result.

THEOREM 3.3. *The value function (2.2) is determined by*

$$V(x, i) = i^{1-\gamma} \left\{ \varphi(z_0) - z_0 \varphi'(z_0) \right\},$$

where $z_0 = \varphi'^{-1}(-x/i)$. Moreover, the function $\varphi(z)$ is given by

$$\varphi(z) = \frac{\gamma}{K(1-\gamma)} z^{1-\frac{1}{\gamma}} + \frac{1}{r_I} z,$$

where the constant K is a Merton constant as

$$K = r + \frac{\beta - r}{\gamma} - \frac{1-\gamma}{2\gamma^2} \theta^2.$$

Notice that since $\varphi'(z)$ in Theorem 3.3 is strictly increasing, there exists one-to-one correspondence between the new variable $z_t \in (0, \infty)$ and wealth process $X_t \in (-I_t/r_I, \infty)$. Now suffice it to show whether the solution $\varphi(z)$ in Theorem 3.3 gives the same value function with those obtained by using dynamic programming or martingale method. By simple calculation, it is confirmed that Theorem 3.3 provides the well-known value function as Merton [11]:

$$V(x, i) = \frac{1}{K^\gamma (1-\gamma)} (x + i/r_I)^{1-\gamma}.$$

Furthermore, the optimal consumption rate c_t^* and investment amount π_t^* are given by

$$(3.4) \qquad c_t^* = (V_x)^{-\frac{1}{\gamma}} = I_t z_t^{*-\frac{1}{\gamma}} = K(X_t + I_t/r_I),$$

$$\pi_t^* = -\frac{\theta V_x}{\sigma_s V_{xx}} - I_t \frac{\sigma_I V_{xi}}{\sigma_s V_{xx}} = -\frac{\sigma_z I_t}{\sigma_s} \varphi''(z_t^*) z_t^* - \frac{\sigma_I I_t}{\sigma_s} \varphi'(z_t^*)$$

$$(3.5) \qquad = \frac{\theta}{\gamma \sigma_s} (X_t + I_t/r_I) - \frac{\sigma_I I_t}{\sigma_s r_I},$$

where z_t^* is the state variable in (3.2) with the initial value $z_0^* = z_0$ in Theorem 3.3.

4. Borrowing constraint

The developed method in Section 3 can be easily applied to the problem with borrowing constraint which allows no borrowing against the agent's future income ($X_t \geq 0$). Thus, the wealth level should always be nonnegative so it is called a nonnegative wealth constraint. Since we presume the wealth process as $X_t = -I_t \varphi'(z_t)$, the constraint implies

$$(4.1) \qquad \varphi'(\bar{z}) = 0, \qquad \varphi''(\bar{z}) = 0,$$

where $\bar{z}$ is the free boundary value which corresponds to the minimum wealth level. The first condition is the value-matching condition and the second condition is the smooth-pasting condition. Therefore, suffice it to solve the ODE (3.3) with the free boundary conditions (4.1) and check Assumption 3.1. Furthermore, this free boundary problem is exactly same with the ODE derived by the martingale method well-established in Dybig and Liu (2010) even thought this approach is totally different mathematically. We summarize the results as follows.

THEOREM 4.1. *When α_+ is the positive real root of the following quadratic equation,*

$$\frac{1}{2}\sigma_z^2\alpha^2 - \left(r_I - \hat{\beta} + \frac{1}{2}\sigma_z^2\right)\alpha - \hat{\beta} = 0,$$

the borrowing constrained agent's value function (2.2) is determined by

$$V(x,i) = i^{1-\gamma}\left\{\varphi(z_0) - z_0\varphi'(z_0)\right\},$$

where $z_0 = \varphi'^{-1}(-x/i)$. Moreover, the function $\varphi(z)$ is given by

$$\varphi(z) = Az^{\alpha_+} + \frac{\gamma}{(1-\gamma)K}z^{1-\frac{1}{\gamma}} + \frac{1}{r_I}z, \quad 0 \le z \le \bar{z}$$

where the free boundary value $\bar{z}$ and the coefficient A are given by

$$\bar{z} = \left(\frac{r_I(\gamma\alpha_+ - \gamma + 1)}{\gamma K(\alpha_+ - 1)}\right),$$

$$A = -\frac{1}{\gamma K\alpha_+(\alpha_+ - 1)}\left(\frac{\gamma K(\alpha_+ - 1)}{r_I(\gamma\alpha_+ - \gamma + 1)}\right)^{\gamma\alpha_+ - \gamma + 1}.$$

Note that since the coefficient A is negative, the function $\varphi'(z)$ is strictly decreasing and satisfies Assumption 3.1. Moreover, the optimal value function is the same with that in Dybvig and Liu [4] so it is confirmed that our method leads to the explicit solution with reduced complexity.

Finally, we want to emphasize that when the agent is not allowed to borrow, the dynamic programming method is hard to characterize the value function containing the condition and the martingale method still needs complicated procedure until the dual value function is derived. However, our method is easy to characterize the nonnegative wealth constraint and also suffice to solve an ODE with two free boundary conditions, which requires an additional term of a general solution to the ODE in (3.3).

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