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TEICHMÜLLER SPACES OF NONORIENTABLE 3-DIMENSIONAL FLAT MANIFOLDS

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Abstract. The various deformation spaces associated with maximal geometric structures on closed oriented 3-manifolds was studied in [2], leaving out the geometry of $\mathbb{R}^3$. In this paper, we study the Weil spaces and Teichmüller spaces of non-orientable 3-dimensional flat Riemannian manifolds. In particular, we find the Teichmüller spaces are homeomorphic to the Euclidean spaces $\mathbb{R}^4$ or $\mathbb{R}^3$ depending on the holonomy group $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ respectively.

1. Introduction

The group of affine motions on the Euclidean space $\mathbb{R}^n$ is $\mathrm{Aff}(n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$.

The group law is

$$h(\mathbf{a}, \mathbf{A}) \cdot (\mathbf{b}, \mathbf{B}) = (\mathbf{a} + \mathbf{A}\mathbf{b}, \mathbf{A}\mathbf{B}),$$

and it acts on $\mathbb{R}^n$ by

$$(\mathbf{a}, \mathbf{A}) \cdot \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{a}$$

for $(\mathbf{a}, A), (\mathbf{b}, B) \in \mathrm{Aff}(n)$ and $\mathbf{x} \in \mathbb{R}^n$. Let $\mathrm{Isom}(\mathbb{R}^n)$ denote the group of isometries of $\mathbb{R}^n$. So

$$\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n) \subset \mathrm{Aff}(n),$$

where $\mathrm{O}(n)$ is the n-dimensional orthogonal group. A subgroup π of $\mathrm{Isom}(\mathbb{R}^n)$ is said to be *crystallographic* if π is compact and discrete. If a crystallographic group π is torsion free, we say π is a *Bieberbach*

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subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. If π is a Bieberbach subgroup of $\mathrm{Isom}(\mathbb{R}^n)$, then the quotient space $\mathbb{R}^n/\pi$ is a Riemannian manifold of sectional curvature $\kappa = 0$. Conversely, a flat closed Riemannian manifold of dimension n is necessarily a quotient of $\mathbb{R}^n$ by a Bieberbach subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ [4].

The following Bieberbach's second theorem says that if two flat Riemannian manifolds are homotopy equivalent, then they are affinely diffeomorphic (see [3]).

THEOREM 1.1 (Bieberbach). *Two crystallographic groups are isomorphic if and only if they are conjugate by an element of the affine group.*

It is known that there are only 10 affine diffeomorphism classes of connected closed 3-dimensional flat manifolds. Six of them are orientable and the others are not. The authors studied the Weil spaces and Teichmüller spaces of the six orientable 3-dimensional flat Riemannian manifolds in [1]. In this paper, those of nonorientable case will be investigated. We use the notation $\mathcal{I}$ for the isometry group $\mathrm{Isom}(\mathbb{R}^3)$ through this paper. So,

$$\mathcal{I} = \mathrm{Isom}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \mathrm{O}(3).$$

2. Preliminaries

For a Bieberbach group π, we define the space of discrete representations of π into $\mathcal{I}$, *the Weil space*, as follows:

$$\mathcal{R}(\pi;\mathcal{I}) = \text{ the space of all injective homomorphisms } \theta \text{ of } \pi \text{ into } \mathcal{I} \text{ such that } \theta(\pi) \text{ is discrete in } \mathcal{I} \text{ and } \mathcal{I}/\theta(\pi) \text{ is compact.}$$

If $\theta, \theta' \in \mathcal{R}(\pi;\mathcal{I})$, then $\mathbb{R}^3/\theta(\pi)$ and $\mathbb{R}^3/\theta'(\pi)$ are affinely diffeomorphic. For $g \in \mathcal{I}$, $\mu(g)$ denotes the conjugation by g. The group $\mathbf{Inn}(\mathcal{I})$ of inner automorphisms of $\mathcal{I}$ acts on the space $\mathcal{R}(\pi,\mathcal{I})$ from the left by

$$\mathbf{Inn}(\mathcal{I}) \times \mathcal{R}(\pi,\mathcal{I}) \to \mathcal{R}(\pi,\mathcal{I}).$$

$$(\mu(g), \theta) \longmapsto \mu(g) \circ \theta$$

The orbit space of this action is called the *Teichmüller space*. That is,

$$\mathcal{T}(\pi, \mathcal{I}) = \mathbf{Inn}(\mathcal{I}) \setminus \mathcal{R}(\pi, \mathcal{I}).$$

If θ and $\theta' \in \mathcal{R}(\pi, \mathcal{I})$ represent the same point in $\mathcal{T}(\pi, \mathcal{I})$, then $\theta' = \mu(g) \circ \theta$ for some $g \in \mathcal{I}$. This implies

$$g \cdot \theta(\alpha)(x) = (g \cdot \theta(\alpha) \cdot g^{-1}) \cdot g(x) = \theta'(\alpha) \cdot g(x) \sim g(x)$$

for all $\alpha \in \pi$. Thus an isometry g of $\mathbb{R}^3$ induces an isometry $\bar{g} : \mathbb{R}^3/\theta(\pi) \to \mathbb{R}^3/\theta'(\pi)$ for which the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{g} & \mathbb{R}^3 \\ \downarrow & & \downarrow \\ \mathbb{R}^3/\theta(\pi) & \xrightarrow{\bar{g}} & \mathbb{R}^3/\theta'(\pi). \end{array}$$

The next theorem, which says that there are only four 3-dimensional nonorientable manifolds, is in [4]. For the convenience we restate here.

THEOREM 2.1. [4] *There are just 4 affine diffeomorphism classes of compact connected nonorientable flat 3-dimensional Riemannian manifolds. They are represented by the manifolds $\mathbb{R}^3/\pi$ where π is one of the 4 groups $\mathfrak{B}_i$ $(1 \le i \le 4)$ given below. Here $\boldsymbol{t}_1$, $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$ are translations by $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ respectively, and $\Phi = \pi/\mathbb{Z}^3$ is the holonomy.*

(1) $\mathfrak{B}_1$ is generated by $\{\boldsymbol{t}_1, \ \boldsymbol{t}_2, \ \boldsymbol{t}_3, \ \epsilon\}$ where $\epsilon^2 = \boldsymbol{t}_1$, $\epsilon \boldsymbol{t}_2 \epsilon^{-1} = \boldsymbol{t}_2$ and $\epsilon \boldsymbol{t}_3 \epsilon^{-1} = \boldsymbol{t}_3^{-1}$; $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal to $\mathbf{a}_3$ while $\epsilon = (\boldsymbol{t}_{\mathbf{a}_1/2}, \ E)$ with $E(\mathbf{a}_1) = \mathbf{a}_1$, $E(\mathbf{a}_2) = \mathbf{a}_2$, $E(\mathbf{a}_3) = -\mathbf{a}_3$.

(2) $\mathfrak{B}_2$ is generated by $\{\boldsymbol{t}_1, \ \boldsymbol{t}_2, \ \boldsymbol{t}_3, \ \epsilon\}$ where $\epsilon^2 = \boldsymbol{t}_1, \epsilon \boldsymbol{t}_2 \epsilon^{-1} = \boldsymbol{t}_2$ and $\epsilon \boldsymbol{t}_3 \epsilon^{-1} = \boldsymbol{t}_1 \boldsymbol{t}_2 \boldsymbol{t}_3^{-1}$; the orthogonal projection of $\mathbf{a}_3$ on the $(\mathbf{a}_1\mathbf{a}_2)$-plane is $\frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2)$, and $\epsilon = (\boldsymbol{t}_{\mathbf{a}_1/2}, E)$ with $E(\mathbf{a}_1) = \mathbf{a}_1, E(\mathbf{a}_2) = \mathbf{a}_2$ and $E(\mathbf{a}_3) = \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3$.

(3) $\mathfrak{B}_3$ *is generated by* $\{\boldsymbol{t}_1,\ \boldsymbol{t}_2,\ \boldsymbol{t}_3,\ \alpha,\ \epsilon\}$ *where* $\alpha^2 = \boldsymbol{t}_1, \epsilon^2 = \boldsymbol{t}_2, \epsilon\alpha\epsilon^{-1} = \boldsymbol{t}_2\alpha, \alpha\boldsymbol{t}_2\alpha^{-1} = \boldsymbol{t}_2^{-1}, \alpha\boldsymbol{t}_3\alpha^{-1} = \boldsymbol{t}_3, \epsilon\boldsymbol{t}_1\epsilon^{-1} = \boldsymbol{t}_1$ *and* $\epsilon\boldsymbol{t}_3\epsilon^{-1} = \boldsymbol{t}_3^{-1}$ *; The* $\mathbf{a}_i$ *are mutually orthogonal and*

$$\alpha = (\boldsymbol{t}_{\mathbf{a}_1/2},\ A) \text{ with } A(\mathbf{a}_1) = \mathbf{a}_1,\ A(\mathbf{a}_2) = -\mathbf{a}_2,\ A(\mathbf{a}_3) = -\mathbf{a}_3,$$

$$\epsilon = (\boldsymbol{t}_{\mathbf{a}_2/2},\ E) \text{ with } E(\mathbf{a}_1) = \mathbf{a}_1,\ E(\mathbf{a}_2) = \mathbf{a}_2,\ E(\mathbf{a}_3) = -\mathbf{a}_3.$$

(4) $\mathfrak{B}_4$ *is generated by* $\{\boldsymbol{t}_1,\ \boldsymbol{t}_2,\ \boldsymbol{t}_3,\ \alpha,\ \epsilon\}$ *where* $\alpha^2 = \boldsymbol{t}_1,\ \epsilon^2 = \boldsymbol{t}_2,\ \epsilon\alpha\epsilon^{-1} = \boldsymbol{t}_2\boldsymbol{t}_3\alpha, \alpha\boldsymbol{t}_2\alpha^{-1} = \boldsymbol{t}_2^{-1},\ \alpha\boldsymbol{t}_3\alpha^{-1} = \boldsymbol{t}_3^{-1},\ \epsilon\boldsymbol{t}_1\epsilon^{-1} = \boldsymbol{t}_1,\ \epsilon\boldsymbol{t}_3\epsilon^{-1} = \boldsymbol{t}^{-1}$*; the* $\mathbf{a}_i$ *are mutually orthogonal and*

$$\alpha = (\boldsymbol{t}_{\mathbf{a}_1/2},\ A) \text{ with } A(\mathbf{a}_1) = \mathbf{a}_1,\ A(\mathbf{a}_2) = -\mathbf{a}_2,\ A(\mathbf{a}_3) = -\mathbf{a}_3,$$

$$\epsilon = (\boldsymbol{t}_{(\mathbf{a}_2+\mathbf{a}_3)/2},\ E) \text{ with } E(\mathbf{a}_1) = \mathbf{a}_1,\ E(\mathbf{a}_2) = \mathbf{a}_2,\ E(\mathbf{a}_3) = -\mathbf{a}_3.$$

Let $X = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$ be a 3×3 matrix of which three column vectors are $\mathbf{x}_1,\ \mathbf{x}_2$ and $\mathbf{x}_3$. The (i, j)-entry of the symmetric matrix X^TX is the inner product $\langle \mathbf{x}_i,\ \mathbf{x}_j \rangle$ of two column vectors $\mathbf{x}_i$ and $\mathbf{x}_j$ of X. In this work the symmetric matrix X^TX is useful because;

LEMMA 2.2. *Let* A *be an orthogonal matrix. For any invertible matrix* X*, the conjugate* XAX^{-1} *of* A *by* X *is orthogonal if and only if* X^TX *and* A *are commutative.*

Proof. The fact that XAX^{-1} is orthogonal means $(XAX^{-1})(XAX^{-1})^T = I$, which is equivalent to $XA(X^TX)^{-1} = X^{-T}A$. Therefore XAX^{-1} is orthogonal if and only if

$$(X^TX)A = A(X^TX).$$

□

Consider a 3×3 non-singular matrix A. Let $\mathcal{X}(A)$ be the space consisting of 3×3 non-singular matrices by which the conjugates of A are orthogonal., i.e.,

$$\mathcal{X}(A) = \{X \in \mathrm{GL}(3, \mathbb{R}) \mid XAX^{-1} \text{ is orthogonal } \}.$$

Note that $\mathcal{X}(A)$ is not a subgroup of $\mathrm{GL}(3, \mathbb{R})$ but we are concerned with the topology on $\mathcal{X}(A)$.

LEMMA 2.3. *If two orthogonal matrices A and B are similar, then $\mathcal{X}(A)$ and $\mathcal{X}(B)$ are homeomorphic.*

Proof. Let P be the 3×3 invertible matrix, with $B = PAP^{-1}$. It is obvious that the correspondence from $X \in \mathcal{X}(A)$ to $XP^{-1} \in \mathcal{X}(B)$ is a homeomorphism. □

3. Main Results

We start with looking at a notation of the topological space obtained from two subgroups H_1 and H_2 of G, i.e.,

$$\mathrm{H}_1 \cdot \mathrm{H}_2 = \{h_1 \cdot h_2 \mid h_1 \in \mathrm{H}_1, \text{ and } h_2 \in \mathrm{H}_2\}.$$

Note that $\mathrm{H}_1 \cdot \mathrm{H}_2$ need not be a subgroup but a subspace of G. Of course H_1 and H_2 may have a nontrivial subgroup in common.

A Bieberbach group π contains a unique maximal normal abelian subgroup $\mathbb{Z}^3$, fitting the following commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \pi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \theta\downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \rtimes O(3) & \longrightarrow & O(3) & \longrightarrow & 1, \end{array} \tag{3.1}$$

where Φ is called the *holonomy group* of π. It is a finite group and $\Phi \to \mathrm{O}(3)$ is injective.

The following theorem says that two 3-dimensional Bieberbach groups with isomorphic holonomies yield the same Weil spaces.

THEOREM 3.1. *Let M be a 3-dimensional nonorientable flat manifold with $\pi_1(M) = \pi$. Then the Weil space $\mathcal{R}(\pi;\mathcal{I})$ is 8-dimensional. Specifically.*

(1) If $\Phi = \mathbb{Z}_2$, then $\mathcal{R}(\pi;\mathcal{I}) = \mathbb{R}^3 \rtimes (O(3) \cdot (\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*))/(\mathbb{R}\boldsymbol{e}_1 \oplus \mathbb{R}\boldsymbol{e}_2) \rtimes \{\mathrm{I}\}$,

(2) If $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{R}(\pi;\mathcal{I}) = \mathbb{R}^3 \rtimes (O(3) \cdot (\mathbb{R}^*)^3)/\mathbb{R}\boldsymbol{e}_1 \rtimes \{\mathrm{I}\}$.

Proof. In each case there are four steps to obtain the Weil space

step 1 Find an embedding θ_0 of π into $\mathcal{I}$. As mentioned in Theorem 1.1, if θ_0 and θ are two embeddings of a Bieberbach group π into $\mathcal{I}$, then their images $\theta_0(\pi)$ and $\theta(\pi)$ are conjugate by an affine motion. That is, there exists an element $\xi \in \mathrm{Aff}(3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R})$ such that $\theta(\pi) = \xi \cdot \theta_0(\pi) \cdot \xi^{-1}$. So,

step 2 Find all members ξ of $\mathrm{Aff}(\mathbb{R}^3)$ which conjugates $\theta_0(\pi)$ into $\mathcal{I}$. Note that this fact depends only on the matrix part of ξ. For a holonomy group $\Phi \subset \mathrm{O}(3)$, let

$$\mathcal{X}(\Phi) = \{X \in \mathrm{GL}(3,\mathbb{R}) | XAX^{-1} \text{ is orthogonal for all } A \in \Phi\}.$$

Observe that $\mathcal{X}(\Phi)$ does not have to be a group. But we need only its 'topological' structure. In fact, the space of all such $\xi \in \mathrm{Aff}(\mathbb{R}^3)$ is

$$\{\ \xi \in \mathrm{Aff}(3) | \theta(\pi) = \xi \cdot \theta_0(\pi) \cdot \xi^{-1}\} = \mathbb{R}^3 \rtimes \mathcal{X}(\Phi).$$

step 3 Find the centralizer $\mathcal{C}_{\mathrm{Aff}(3)}(\theta_0(\pi))$ of $\theta_0(\pi)$ in the group $\mathrm{Aff}(3)$. The action of Φ on $\mathbb{Z}^3$ is induced from the exactness of the top row of the above diagram (3.1). It is known that the centralizer $\mathcal{C}(\theta_0(\pi))$ is the fixed point set $(\mathbb{R}^3)^\Phi$ of the Φ action on $\mathbb{R}^3$. It is a normal subgroup of $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$. For $\xi \in \mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$ and $\zeta \in \mathcal{C}(\theta_0(\pi))$,

ξ and $\xi \cdot \zeta$ yield the same representation, because $\mu(\xi \cdot \zeta)(\theta_0(\alpha)) = \mu(\xi)\mu(\zeta)(\theta_0(\alpha)) = \mu(\xi)(\theta_0(\alpha))$.

step 4 Factor out $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$ by $\mathcal{C}(\theta_0(\pi)) = (\mathbb{R}^3)^\Phi$. The space of representations is thus the orbit space

$$\mathcal{R}(\pi; \mathcal{I}) = \mathbb{R}^3 \rtimes \mathcal{X}(\Phi)/(\mathbb{R}^3)^\Phi.$$

(1) Case of $\Phi = \mathbb{Z}_2$. Take the embedding $\theta_0 : \pi \to \mathcal{I}$ as a homomorphism defined by

$$\theta_0(\mathbf{t}_i) = (\mathbf{e}_i,\ I) \text{ for } 1 \leq i \leq 2,$$

$$\theta_0(\epsilon) = (\frac{1}{2}\mathbf{e}_1,\ E) \text{ where } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

and if π is isomorphic to $\mathfrak{B}_1$ of Theorem 2.1 then

$$\theta_0(\mathbf{t}_3) = (\mathbf{e}_3,\ I),$$

and if π is isomorphic to $\mathfrak{B}_2$ of Theorem 2.1 then

$$\theta_0(\mathbf{t}_3) = (\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2) + \mathbf{e}_3,\ I).$$

The defining condition $X \in \mathcal{X}(\Phi)$ is $XAX^{-1} \in \mathrm{O}(3)$. It is equivalent to $(X^TX)E = E(X^TX)$ by Lemma 2.2. This implies that the third column vector $\mathbf{x}_3$ of X is orthogonal to the other column vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of X. Hence the space

$$\begin{aligned} \mathcal{X}(\Phi) = &\ \{X = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] \in \mathrm{GL}(3, \mathbb{R}) \mid \mathbf{x}_1 \perp \mathbf{x}_3 \text{ and } \mathbf{x}_2 \perp \mathbf{x}_3\} \\ &= \mathrm{O}(3) \cdot \left\{ \begin{bmatrix} A & O \\ O & b \end{bmatrix} \mid A \in \mathrm{GL}(2, \mathbb{R}) \text{ and } b \in \mathbb{R}^* \right\} \\ &= \mathrm{O}(3) \cdot (\mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^*). \end{aligned}$$

where $\mathbb{R}^*$ means the set of all non-zero real numbers. Note that a 3-dimensional space $\mathrm{O}(3)$ and a 5-dimensional space $\mathrm{GL}(2, \mathbb{R}) \times \mathbb{R}^*$ intersects the common space $\mathrm{O}(2) \times \mathbb{Z}_2$ which is 1-dimensional, and

$\mathcal{X}(\Phi)$ has 4-components. And so $\mathrm{O}(3)\cdot(\mathrm{GL}(2,\mathbb{R})\times\mathbb{R}^*)$ is 7-dimensional. A brief computation shows that the centralizer is given by

$$\begin{aligned}(\mathbb{R}^3)^\Phi &= \{(\mathbf{c},\ I) \in \mathrm{Aff}(3) \mid \mathbf{c} = [*\ *\ 0]^T\}\\ &= \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2.\end{aligned}$$

This concludes the result (1).

(2) Case of $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let's take an embedding $\theta_0 : \pi \to \mathcal{I}$ as follows:

$$\theta_0(\mathbf{t}_i) = (\mathbf{e}_i,\ I) \quad (1 \le i \le 3),$$

$$\theta_0(\alpha) = (\frac{1}{2}\mathbf{e}_1,\ A),$$

and if π is isomorphic to $\mathfrak{B}_3$ of Theorem 2.1 then

$$\theta_0(\epsilon) = (\frac{1}{2}\mathbf{e}_2,\ E),$$

and if π is isomorphic to $\mathfrak{B}_4$ of Theorem 2.1 then

$$\theta_0(\epsilon) = (\frac{1}{2}(\mathbf{e}_2 + \mathbf{e}_3),\ E),$$

where $A = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ and $E = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$. We look for all matrices X

such that XAX^{-1} and XEX^{-1} are orthogonal. It is equivalent to saying that X^TX is diagonal. Hence

$$\begin{aligned}\mathcal{X}(\Phi) &= \{X \in \mathrm{GL}(3,\mathbb{R}) \mid \mathbf{x}_i \perp \mathbf{x}_j \text{ if } i \neq j\}\\ &= \mathrm{O}(3)\cdot(\mathbb{R}^*)^3.\end{aligned}$$

The 3-dimensional spaces $\mathrm{O}(3)$ and $(\mathbb{R}^*)^3$ have intersection $(\mathbb{Z}_2)^3$, consisting of all diagonal matrices with entries ± 1. Since this space is

0-dimensional, $\mathcal{X}(\Phi)$ is 6-dimensional. Clearly we get the centralizer

$$\begin{aligned}(\mathbb{R}^3)^\Phi &= \{(\mathbf{c},\ I) \in \mathrm{Aff}(3) \mid \mathbf{c} = [*\ 0\ 0]^T\} \\ &= \mathbb{R}\mathbf{e}_1,\end{aligned}$$

and the Weil space

$$\mathcal{R}(\pi;\mathcal{I}) = \mathbb{R}^3 \rtimes (\mathrm{O}(3)\cdot(\mathbb{R}^*)^3)/\mathbb{R}\mathbf{e}_1 \rtimes \{I\}.$$

□

Remark 0.1. *In each case of the above theorem, the (right) action of* $(\mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2) \rtimes \{I\} \cong \mathbb{R}^2$ *or* $\mathbb{R}\mathbf{e}_1 \rtimes \{I\} \cong \mathbb{R}$ *on* $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$ *is twisted. In other words, one cannot write the orbit space as* $(\mathbb{R}^3 \rtimes \mathcal{X}(\Phi))/\mathbb{R}^2 \rtimes \{I\} \approx \mathbb{R} \rtimes \mathcal{X}(\Phi)$ *or* $(\mathbb{R}^3 \rtimes \mathcal{X}(\Phi))/\mathbb{R} \rtimes \{I\} \approx \mathbb{R}^2 \rtimes \mathcal{X}(\Phi)$*. However the action is free and proper so that the orbit space is a manifold.*

Finally we show that the Teichmüller space is homeomorphic to the Euclidean space $\mathbb{R}^4$ or $\mathbb{R}^3$ depending on the holonomy group $\mathbb{Z}_2$ or $\mathbb{Z}^2 \times \mathbb{Z}^2$ respectively.

Theorem 3.2. *Let M be a 3-dimensional nonorientable flat manifold with $\pi(M) = \pi$. Then the Teichmüller spaces are as follow:*

(1) If $\Phi = \mathbb{Z}_2$, then $\mathcal{T}(\pi;\mathcal{I}) = (\mathrm{O}(2) \setminus \mathrm{GL}(2,\mathbb{R})) \times \mathbb{R}^+ \approx \mathbb{R}^3 \times \mathbb{R}^+ \approx \mathbb{R}^4$.

(2) If $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{T}(\pi;\mathcal{I}) = (\mathbb{Z}_2)^3 \setminus (\mathbb{R}^*)^3 = (\mathbb{R}^+)^3 \approx \mathbb{R}^3$.

Proof. The isometry group $\mathcal{I} = \mathbb{R}^3 \rtimes O(3)$ acts on $\mathcal{R}(\pi,\mathcal{I})$ on the left by conjugation, and the orbit space is the Teichmüller space of π. On the space $\mathbb{R}^3 \rtimes \mathcal{X}(\Phi)$ level the action is just a multiplication from the left. From $\mathbb{R}^3 \rtimes \{I\} \subset \mathbb{R}^3 \rtimes \mathrm{O}(3)$, every orbit must contain whole $\mathbb{R}^3$. Thus, the Teichmüller space is simply

$$\mathrm{O}(3) \setminus \mathcal{X}(\Phi).$$

For a general fact, recall that $\mathrm{O}(n)$ is a maximal compact subgroup of $\mathrm{GL}(n,\mathbb{R})$, and $\mathrm{O}(n)\setminus\mathrm{GL}(n,\mathbb{R})\approx\mathbb{R}^{\frac{n(n+1)}{2}}$. Thus we have the following theorem. □

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