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QUASI-ANOSOV DIFFEOMORPHISMS AND VARIOUS SHADOWING PROPERTIES

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Abstract. In this paper, we show that if a quasi-Anosov diffeomorphism has the various types of shadowing property then it is Anosov.

1. Introduction

Let M be a closed smooth Riemannian manifold and let $f : M \to M$ be a diffeomorphism. Denote by Diff(M) the set of all diffeomorphisms of M endowed with the C^1 topology.

Let $f \in$ Diff(M) and let Λ be a closed f-invariant set. We say that Λ is *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \le C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. If $\Lambda = M$ then we say that f is *Anosov*. We say that f is *quasi-Anosov* if for every $0 \ne v \in TM$ then set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded.

Note that a quasi-Ansov diffeomophism f is not Anosov (see [3]). But if dim$M = 2$ then a Anosov diffeomorphism is a quasi-Anosov diffeomorphism (see [3]). A point $p \in M$ is said to be *periodic* if there is $n > 0$ such that $f^n(p) = p$. Denote by $P(f)$ the set of all periodic points of f. A point $x \in M$ is said to be *non-wandering* if for any neighborhood U of x there is $n > 0$ such that $f^n(U) \cap U \ne \emptyset$. Denote by $\Omega(f)$ the set of all non-wandering points of f. It is clear that $P(f) \subset \Omega(f)$. We

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say that f satisfies *Axiom A* if $\Omega(f) = \overline{P(f)}$ is hyperbolic. We say that f is *structurally stable* if there is a neighborhood $\mathcal{U}(f) \subset \text{Diff}(M)$ such that for every $g \in \mathcal{U}(f)$, there is a homeomorphism $h : M \to M$ such that $f \circ h = h \circ g$. We define the stable set of x as follows: $W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}$, and the unstable set of x as follows: $W^u(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\}$. We say that an Axiom A diffeomorphism f satisfies the *transversality condition* if for any $x \in M$, $T_xM = T_xW^s(x) + T_xW^u(x)$. In [8], Mañé proved that a diffeomorphism f is Anosov if and only if f is quasi-Anosov and satisfies the transversality condition if and only if f is quasi-Anosov and structurally stable.

For $\delta > 0$, a sequence of points $\{x_i\}_{i\in\mathbb{Z}}$ in M is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$, there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Mañé [8] proved that a quasi-Anosov diffeomorphism f if and only if an Axiom A diffeomorphisms satisfying $T_xW^s(x) \cap T_xW^u(x) = \{0_x\}$ for every $x \in M$.

Sakai [11] proved that every quasi-Anosov diffeomorphism with shadowing property is Anosov. From the results, we consider that if a qausi-Anosov diffeomorphism with the various shadowing properties(asymptotic average shadowing, average shadowing, ergodic shadowing) then it is Anosov.

The asymptotic average shadowing property introduced by Gu [5]. A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called an *asymptotic average pseudo orbit* of f if

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f(x_i), x_{i+1}) = 0.$$

A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be asymptotic average shadowed in average by the point z in M if

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) = 0.$$

We say that f has the *asymptotic average shadowing property* if every asymptotic average pseudo orbit of f can be asymptotic average shadowed in average by some point in M. The average shadowing property was introduced by Blank [1]. For $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in M is called a *δ-average pseudo orbit* of f if there is $N(\delta) > 0$ such that for all $n \geq N, k \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the *average shadowing property* if for any $\epsilon > 0$ there is a $\delta > 0$ such that every δ-average pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ is ϵ-shadowed in average by some $z \in M$, that is,

$$\limsup_{n\to\infty} \frac{1}{2n}\sum_{i=-n}^{n-1} d(f^i(z), x_i) < \epsilon.$$

The notion of ergodic shadowing property for continuous onto maps over compact metric spaces was defined by Fakhari and Ghane in [2]. For any $\delta > 0$, a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is δ-*ergodic pseudo orbit* of f if for $Np_n^+(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} \cap \{0, 1, \ldots, n-1\}$, and $Np_n^-(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} \cap \{0, -1, \ldots, -n+1\}$,

$$\lim_{n\to\infty} \frac{\#Np_n^+(\xi, f, \delta)}{n} = 0 \text{ and } \lim_{n\to\infty} \frac{\#Np_n^-(\xi, f, \delta)}{n} = 0.$$

We say that f has the *ergodic shadowing property* if for any $\epsilon > 0$, there is a $\delta > 0$ such that every δ-ergodic pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ of f there is a point $z \in M$ such that for $Ns_n^+(\xi, f, z, \epsilon) = \{i : d(f^i(z), x_i) \geq \epsilon\} \cap \{0, 1, \ldots, n-1\}$, and $Ns_n^-(\xi, f, z, \epsilon) = \{i : d(f^i(z), x_i) \geq \epsilon\} \cap \{0, -1, \ldots, -n+1\}$,

$$\lim_{n\to\infty} \frac{\#Ns_n^+(\xi, f, z, \epsilon)}{N} = 0 \text{ and } \lim_{n\to\infty} \frac{\#Ns_n^-(\xi, f, z, \epsilon)}{N} = 0.$$

Then we have the following which is a main theorem in this paper.

THEOREM 1.1. *Let $f \in \mathrm{Diff}(M)$ be quasi-Anosov. If any of the following statements hold:*

(a) *f has the asymptotic average shadowing property,*
(b) *f has the average shadowing property,*
(c) *f has the ergodic shadowing property,*

then f is Anosov.

2. Proof of Theorem 1.1

Let M be as before and let $f \in \mathrm{Diff}(M)$. For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a finite δ-pseudo orbit $\{x_i\}_{i=0}^n (n \geq 1)$ of f such that $x_0 = x$ and $x_n = y$. For any $x, y \in \Lambda$, we write $x \rightsquigarrow_\Lambda y$

if $x \rightsquigarrow y$ and $\{x_i\}_{i=0}^n \subset \Lambda(n \geq 1)$. We say that the set $\mathcal{C}(f)$ is *chain transitive* if for any $x, y \in \mathcal{C}(f)$, $x \rightsquigarrow_{\mathcal{C}(f)} y$. If $\mathcal{C}(f) = M$ then f is said to be *chain transitive.*

We say that f is *robustly chain transitive* if there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of $\mathcal{C}(f)$ such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is chain transitive, where $\Lambda_g(U)$ is the continuation of $\mathcal{C}(f)$. Lee [7] proved that for any periodic points $p, q \in \mathcal{C}(f)$ if $\mathcal{C}(f)$ is robustly chain transitive and index(p) = index(q) then it is hyperbolic, where index$(p) = \dim W^s(p)$. Lee and Park [6] proved that C^1 generically, if a diffeomorphism f has the asymptotic average, or average shadowing property and $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$ then it is hyperbolic. For that, chain transitive diffeomorphisms and various types of shadowing properties are related to the hyperbolicity. The set $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $R(f)$. It is easy to see that the set is closed and $f(R(f)) = R(f)$. The relation $\leftrightsquigarrow$ induces an equivalence relation on $R(f)$ whose equivalence classes are called *chain component* of f and is denoted by C_f. In general, the chain component is a closed and invariant set. Note that a chain component C_f is a maximal chain transitive set.

LEMMA 2.1. *If f is chain transitive then the chain recurrence set $R(f)$ is M.*

Proof. Clearly, $R(f) \subset M$. Thus we show that $M \subset R(f)$. Note that a chain component C_f is a maximal chain transitive. Since f is chain transitive, we know that M is contained in a chain component C_f. Since the chain component $C_f \subset R(f)$, we have $M \subset R(f)$. Thus if f is chain transitive then $R(f) = M$. □

LEMMA 2.2. *Let $f \in \mathrm{Diff}(M)$ be Ω-stable. If f is chain transitive then it is Anosov.*

Proof. Suppose that f is Ω-stable. Note that if f is Ω-stable then f satisfies Axiom A without cycles (see [9]). Since f satisfies Axiom A, we know that $\Omega(f) = \overline{P(f)}$ is hyperbolic. The result of Franks and Selgrade [4, Theorem A], that is, the chain recurrence set $R(f)$ is hyperbolic if and only if it is Ω-stable. Since f is chain transitive, by Lemma 2.1, $R(f) = M$. Since f is Ω-stable, $\Omega(f) = R(f) = M$ is hyperbolic. Thus f is Anosov. □

Gu [5, Theorem 3.1] proved that if a diffeomorphism f has the asymptotic average shadowing property then it is chain transitive, Park and

Zhang [10, Theorem 3.4] proved that if a diffeomorphism f has the average shadowing property then it is chain transitive. Fakhari and Ghane[2, Lemma 3.1] proved that if a diffeomorphism f has the ergodic shadowing property then it is chain transitive. From the above results, we rewrite as the following.

LEMMA 2.3. *If f has the asymptotic average, average, ergodic shadowing property then it is chain transitive.*

Proof of Theorem 1.1. Let f be a quasi-Anosov diffeomorphism. Suppose that a diffeomorphism f has the asymptotic average, average, or ergodic shadowing property. By Lemma 2.3, it is chain transitive. Since f satisfies Axiom A, by Lemma 2.2 f is Anosov. □

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