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BOUNDEDNESS FOR NONLINEAR PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

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Abstract. This paper shows that the solutions to the nonlinear perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t)),$$

have bounded properties. To show these properties, we impose conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), T_1 y(s))ds$, $h(t, y(t), T_2 y(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t,y)$ using the notion of h-stability.

1. Introduction

Pachpatte[16,17] investigated the stability, boundedness, and the asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term g and on the operator T. The purpose of this paper is to investigate bounds for solutions of the nonlinear differential systems further allowing more general perturbations that were previously allowed using the notion of h-stability.

The notion of h-stability (hS) was introduced by Pinto [18,19] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Choi, Ryu [7] and Choi, Koo [8] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [10,11,12] and

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Im et al. [5,6,14] studied the boundedness of solutions for the perturbed differential systems.

2. preliminaries

In this paper we study bounds of solutions for a class of the nonlinear perturbed differential systems of the form

(2.1)
$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t)),\ y(t_0) = y_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g, h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) = 0$, $g(t,0,0) = h(t,0,0) = 0$, $\mathbb{R}^n$ is the Euclidean n-space and $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ are continuous operators. We consider nonlinear unperturbed differential system of (2.1)

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.2}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t,0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \le 1} |Ax|$.

We let $x(t, t_0, x_0)$ denote the unique solution of (2.2) passing through (t_0, x_0), existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.2) and around $x(t)$, respectively,

$$v'(t) = f_x(t,0)v(t),\ v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t),\ z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

We introduce some notions[19] and results to be used in this paper.

Definition 2.1. The system (2.2) (the zero solution $x = 0$ of (2.2)) is called an *h-system* if there exist a constant $c \ge 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \le c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough $\big($here $h(t)^{-1} = \frac{1}{h(t)}\big)$.

Definition 2.2. The system (2.2) (the zero solution $x = 0$ of (2.2)) is called (hS) *h-stable* if there exists $\delta > 0$ such that (2.2) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [9].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞*-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)| dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{2.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [9, 13].

Lemma 2.4. [19] *The linear system*

$$x' = A(t)x,\ x(t_0) = x_0, \tag{2.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\Phi(t, t_0, x_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{2.7}$$

for $t \geq t_0 \geq 0$, where $\Phi(t, t_0, x_0)$ is a fundamental matrix of (2.6).

We need Alekseev formula to compare between the solutions of (2.2) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{2.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following result is due to Alekseev [1].

Lemma 2.5. [2] *Let x and y be a solution of (2.2) and (2.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t,t_0,y_0) \in \mathbb{R}^n$, $y(t,t_0,y_0) \in \mathbb{R}^n$,*

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^t \Phi(t,s,y(s))\, g(s,y(s))\, ds.$$

Theorem 2.6. [7] *If the zero solution of (2.2) is hS, then the zero solution of (2.3) is hS.*

Theorem 2.7. [8] *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.3) is hS, then the solution $z = 0$ of (2.4) is hS.*

Lemma 2.8. (*Bihari – type inequality*) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda(s) w(u(s)) ds, \; t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^t \lambda(s) ds\Big], \; t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^t \lambda(s) ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 2.9. [3] *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$,*

$$\begin{aligned} u(t) \leq c \;&+\; \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) w(u(s)) ds \\ &+\; \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) u(\tau) d\tau ds \\ &+\; \int_{t_0}^t \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) w(u(\tau)) d\tau ds, \; 0 \leq t_0 \leq t. \end{aligned}$$

Then

$$\begin{aligned} u(t) \leq W^{-1}\Big[W(c) &+ \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \\ &+ \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds\Big], \end{aligned}$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau + \lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.10. [4] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$,

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t}\lambda_1(s)u(s)ds + \int_{t_0}^{t}\lambda_2(s)w(u(s))ds \\ &+ \int_{t_0}^{t}\lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)w(u(\tau))d\tau ds \\ &+ \int_{t_0}^{t}\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)w(u(\tau))d\tau ds, \ \ 0 \leq t_0 \leq t. \end{aligned}$$

Then

$$\begin{aligned} u(t) \leq W^{-1}\Big[W(c) + \textstyle\int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) \\ +\lambda_3(s)\textstyle\int_{t_0}^{s}\lambda_4(\tau)d\tau + \lambda_5(s)\textstyle\int_{t_0}^{s}\lambda_6(\tau)d\tau)ds\Big], \end{aligned}$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau + \lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.11. [11] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$,

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t}\lambda_1(s)w(u(s))ds + \int_{t_0}^{t}\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)u(\tau) + \lambda_4(\tau)w(u(\tau)) \\ &+ \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)u(r)dr)d\tau ds + \int_{t_0}^{t}\lambda_7(s)\int_{t_0}^{s}\lambda_8(\tau)w(u(\tau))d\tau ds. \end{aligned}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)+\lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr)d\tau + \lambda_7(s)\int_{t_0}^{s}\lambda_8(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)+\lambda_4(\tau) +\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr)d\tau + \lambda_7(s)\int_{t_0}^{s}\lambda_8(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Corollary 2.12. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t}\lambda_1(s)w(u(s))ds + \int_{t_0}^{t}\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)u(\tau)+\lambda_4(\tau)w(u(\tau)) +\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)u(r)dr)d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) +\lambda_4(\tau)+\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)+\lambda_4(\tau) +\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr)d\tau \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 2.13. [12] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t}\lambda_1(s)w(u(s))ds + \int_{t_0}^{t}\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)u(\tau)+\lambda_4(\tau)w(u(\tau)) +\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)w(u(r))dr)d\tau ds + \int_{t_0}^{t}\lambda_7(s)\int_{t_0}^{s}\lambda_8(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)$$
$$+\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr)d\tau + \lambda_7(s)\int_{t_0}^{s}\lambda_8(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)$$
$$+\lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr)d\tau + \lambda_7(s)\int_{t_0}^{s}\lambda_8(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Corollary 2.14. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t}\lambda_1(s)w(u(s))ds + \int_{t_0}^{t}\lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)u(\tau)$$
$$+\lambda_4(\tau)w(u(\tau)) + \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)w(u(r))dr)d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)$$
$$+\lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr)d\tau\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau)$$
$$+\lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr)d\tau \in \mathrm{domW}^{-1}\Big\}.$$

3. Main results

In this section, we investigate boundedness for solutions of the nonlinear perturbed differential systems via t_∞-similarity.

To obtain the bounded result, the following assumptions are needed:

(*H1*) $f_x(t,0)$ is t_∞-similar to $f_x(t, x(t,t_0,x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.

(*H2*) The solution $x = 0$ of (1.1) is hS with the increasing function h.

(*H3*) $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$.

THEOREM 3.1. *Let $a, b, c, d, k \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3) and g in (2.1) satisfies*

$$\begin{aligned} |g(t,y,T_1y)| &\le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, \\ |T_1y(t)| &\le b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} |h(t,y(t),T_2y(t))| &\le \int_{t_0}^{t} c(s)|y(s)|ds + |T_2y(t)|, \\ |T_2y(t)| &\le d(t)w(|y(t)|), \end{aligned} \tag{3.2}$$

where $a, b, c, d, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are continuous operators. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.1) is bounded on on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}[d(s) + \int_{t_0}^{s}(a(\tau) + b(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau]ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}[d(s) + \int_{t_0}^{s}(a(\tau) + b(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau]ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (2.1), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.2) is hS, the solution $v = 0$ of (2.3) is hS. Therefore, from (*H1*), by Theorem 2.7, the solution $z = 0$ of (2.4) is hS. Applying the nonlinear variation of constants formula due to Lemma 2.5, Lemma 2.4 together with (3.1) and (3.2), we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))|\big(\int_{t_0}^{s} |g(\tau, y(\tau), T_1y(\tau))|d\tau \\ &\qquad + |h(s, y(s), T_2y(s))|\big)ds \\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)h(s)^{-1}\Big(d(s)w(|y(s)|) \\ &\qquad + \int_{t_0}^{s}((a(\tau) + c(\tau))|y(\tau)| + b(\tau)w(|y(\tau)|) \\ &\qquad + b(\tau)\int_{t_0}^{\tau} k(r)w(|y(r)|)dr)d\tau\Big)ds. \end{aligned}$$

By the assumptions (*H2*) and (*H3*), we obtain

$$
\begin{aligned}
|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \textstyle\int_{t_0}^t c_2 h(t)\Big(d(s)w(\tfrac{|y(s)|}{h(s)})\\
+ \textstyle\int_{t_0}^s((a(\tau)+c(\tau))\tfrac{|y(\tau)|}{h(\tau)} + b(\tau)w(\tfrac{|y(\tau)|}{h(\tau)})\\
+b(\tau)\textstyle\int_{t_0}^\tau k(r)w(\tfrac{|y(r)|}{h(r)})dr)d\tau\Big)ds.
\end{aligned}
$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Corollary 2.14, we have

$$
\begin{aligned}
|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\textstyle\int_{t_0}^t[d(s) + \int_{t_0}^s(a(\tau)+b(\tau)+c(\tau)\\
+b(\tau)\textstyle\int_{t_0}^\tau k(r)dr)d\tau]ds\Big]
\end{aligned}
$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and so the proof is complete. □

Remark 3.2. Letting $c(t) = d(t) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.5 in [10].

Theorem 3.3. *Let $a, b, c, d, k, q \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.1) satisfies*

$$
\begin{aligned}
(3.3)\qquad \textstyle\int_{t_0}^t |g(s,y(s),T_1y(s))|ds \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|,\\
|T_1y(t)| \le b(t)\textstyle\int_{t_0}^t k(s)|y(s)|ds
\end{aligned}
$$

and

$$
\begin{aligned}
(3.4)\qquad |h(t,y(t),T_2y(t))| \le b(t)\textstyle\int_{t_0}^t c(s)|y(s)|ds + |T_2y(t)|,\\
|T_2y(t)| \le d(t)\textstyle\int_{t_0}^t q(s)w(|y(s)|)ds
\end{aligned}
$$

where $a, b, c, d, k, q, w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are continuous operators. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.1) is bounded on $[t_0, \infty)$ and it satisfies

$$
\begin{aligned}
|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\textstyle\int_{t_0}^t(a(s)+b(s)+c(s)\\
+b(s)\textstyle\int_{t_0}^s k(\tau)d\tau + d(s)\int_{t_0}^s q(\tau)d\tau)ds\Big],
\end{aligned}
$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 2.8, and

$$
\begin{aligned}
b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\textstyle\int_{t_0}^t(a(s)+b(s)+c(s)\\
+b(s)\textstyle\int_{t_0}^s k(\tau)d\tau + d(s)\int_{t_0}^s q(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.
\end{aligned}
$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (2.1), respectively. By the same argument as in the proof in Theorem 3.1, the solution $z = 0$ of (2.4) is hS. Using Lemma 2.4, the nonlinear variation of constants formula due to Lemma 2.5, together with (3.3) and (3.4), we have

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2 h(t)h(s)^{-1}\Big(a(s)|y(s)| + b(s)w(|y(s)|) \\ +b(s)\int_{t_0}^s (c(\tau)+k(\tau))|y(\tau)|d\tau + d(s)\int_{t_0}^s q(\tau)w(|y(\tau)|)d\tau\Big)ds.$$

It follows from (*H2*) and (*H3*) that

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2 h(t)\Big(a(s)\tfrac{|y(s)|}{h(s)} + b(s)w(\tfrac{|y(s)|}{h(s)}) \\ +b(s)\int_{t_0}^s (c(\tau)+k(\tau))\tfrac{|y(\tau)|}{h(\tau)}d\tau + d(s)\int_{t_0}^s q(\tau)w(\tfrac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.9, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t [a(s)+b(s)+c(s) \\ +b(s)\int_{t_0}^s k(\tau)d\tau + d(s)\int_{t_0}^s q(\tau)d\tau]ds\Big],$$

where $c = c_1|y_0|h(t)\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

Remark 3.4. Letting $c(t) = d(t) = 0$ in Theorem 3.3, we obtain the same result as that of Theorem 3.3 in [10].

Theorem 3.5. *Let $a, b, c, d, k \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.1) satisfies*

$$(3.5)\qquad \begin{aligned} |g(t, y, T_1 y)| &\leq a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|, \\ |T_1 y(t)| &\leq b(t)\int_{t_0}^t k(s)|y(s)|ds \end{aligned}$$

and

$$(3.6)\qquad \begin{aligned} |h(t, y(t), T_2 y(t))| &\leq \int_{t_0}^t c(s)|y(s)|ds + |T_2 y(t)|, \\ |T_2 y(t)| &\leq d(t)w(|y(t)|), \end{aligned}$$

where $a, b, c, d, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are continuous operators. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.1) is bounded on *on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} [\int_{t_0}^{s} (a(\tau) + b(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau + d(s) \int_{t_0}^{\tau} q(\tau)d\tau]ds\Big],$$

where W, W^{-1} *are the same functions as in Lemma 2.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2 \int_{t_0}^{t} [\int_{t_0}^{s} (a(\tau) + b(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau + c(s) \int_{t_0}^{\tau} q(\tau)d\tau]ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (2.1), respectively. By the same argument as in the proof in Theorem 3.1, the solution $z = 0$ of (2.4) is hS. Applying Lemma 2.4, the nonlinear variation of constants formula due to Lemma 2.5, together with (3.5) and (3.6), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau), T_1 y(s))|d\tau + |h(s, y(s), T_2 y(s))|)ds \\ &\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)h(s)^{-1}\Big(d(s)w(|y(s)|) + \int_{t_0}^{s} ((a(\tau) + c(\tau))|y(\tau)| + b(\tau)w(|y(\tau)|) \\ &\quad + b(\tau) \int_{t_0}^{\tau} k(r)|y(r)|dr)d\tau\Big)ds.\end{aligned}$$

By the assumptions (H2) and (H3), we obtain

$$\begin{aligned}|y(t)| &\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(d(s)w(\tfrac{|y(s)|}{h(s)}) + b(\tau)w(\tfrac{|y(\tau)|}{h(\tau)}) + \int_{t_0}^{s} ((a(\tau) + c(\tau))\tfrac{|y(\tau)|}{h(\tau)} \\ &\quad + b(\tau) \int_{t_0}^{\tau} k(r)\tfrac{|y(r)|}{h(r)}dr)d\tau\Big)ds.\end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Corollary 2.12, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} [\int_{t_0}^{s} (a(\tau) + b(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau + d(s) \int_{t_0}^{\tau} q(\tau)d\tau]ds\Big]$$

where $c = c_1|y_0|\, h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and so the proof is complete. □

Remark 3.6. Letting $c(t) = d(t) = 0$ in Theorem 3.5, we obtain the same result as that of Theorem 3.1 in [10].

THEOREM 3.7. *Let $a, b, c, d, k, q \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (2.1) satisfies*

$$(3.7) \quad \begin{aligned} \int_{t_0}^{t} |g(s, y(s), T_1 y(s))| ds &\leq a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|, \\ |T_1 y(t)| &\leq b(t) \int_{t_0}^{t} k(s) w(|y(s)|) ds \end{aligned}$$

and

$$(3.8) \quad \begin{aligned} |h(t, y(t), T_2 y(t))| &\leq c(t) \int_{t_0}^{t} q(s) w(|y(s)|) ds + |T_2 y(t)|, \\ |T_2 y(t)| &\leq d(t) w(|y(t)|) \end{aligned}$$

where $a, b, c, d, k, q, w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are continuous operators. Then, any solution $y(t) = y(t, t_0, y_0)$ of (2.1) is bounded on $[t_0, \infty)$ and it satisfies

$$\begin{aligned} |y(t)| \leq h(t) W^{-1} \Big[W(c) + c_2 \textstyle\int_{t_0}^{t} (a(s) + b(s) + c(s) \\ + b(s) \int_{t_0}^{s} k(\tau) d\tau + d(s) \int_{t_0}^{s} q(\tau) d\tau) ds \Big], \end{aligned}$$

where $t_0 \leq t < b_1$, W, W^{-1} are the same functions as in Lemma 2.8, and

$$\begin{aligned} b_1 = \sup \Big\{ t \geq t_0 : W(c) + c_2 \textstyle\int_{t_0}^{t} (a(s) + b(s) + c(s) \\ + b(s) \int_{t_0}^{s} k(\tau) d\tau + d(s) \int_{t_0}^{s} q(\tau) d\tau) ds \in \mathrm{domW}^{-1} \Big\}. \end{aligned}$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (2.1), respectively. By the same argument as in the proof in Theorem 2.2, the solution $z = 0$ of (2.4) is hS. Using Lemma 2.4, the nonlinear variation of constants formula due to Lemma 2.5, together with (3.7) and (3.8), we have

$$\begin{aligned} |y(t)| \leq c_1 |y_0| h(t)\, h(t_0)^{-1} + \textstyle\int_{t_0}^{t} c_2 h(t) h(s)^{-1} \Big(a(s)|y(s)| \\ + (b(s) + d(s)) w(|y(s)|) + b(s) \textstyle\int_{t_0}^{s} k(\tau) w(|y(\tau)|) d\tau \\ + c(s) \textstyle\int_{t_0}^{s} q(\tau) w(|y(\tau)|) d\tau \Big) ds. \end{aligned}$$

It follows from (H2) and (H3) that

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(a(s)\frac{|y(s)|}{h(s)} + (b(s)+d(s))w(\frac{|y(s)|}{h(s)}) + b(s)\int_{t_0}^{s} k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau + c(s)\int_{t_0}^{s} q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.10, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}[a(s)+b(s)+c(s) + b(s)\int_{t_0}^{s} k(\tau)d\tau + d(s)\int_{t_0}^{s} q(\tau)d\tau]ds\Big],$$

where $c = c_1|y_0|h(t)\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

Remark 3.8. Letting $c(t) = d(t) = 0$ in Theorem 3.7, we obtain the same result as that of Theorem 3.7 in [10].

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