

SOME GENERALIZATIONS OF WEAKLY M -SEMI-CONTINUOUS AND WEAKLY M -PRECONTINUOUS FUNCTIONS

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ABSTRACT. As a generalization of (i, j) -weakly m -continuous functions [43], we introduce the notion of weakly $M(i, j)$ -continuous functions and obtain many characterizations and some properties of the functions. We show that the function is a unified form of some functions between m -spaces and certain kinds of weakly continuous functions in bitopological spaces.

1. Introduction

Semi-open sets, preopen sets, α -open sets and β -open sets play an important role in the researching of generalizations of continuity in topological spaces and bitopological spaces. By using these sets many authors introduced and studied various types of modifications of continuity in topological spaces and bitopological spaces. Khedr [18] and the present authors [42], [46], [51] introduced and studied weakly semi-continuous functions and weakly precontinuous functions in bitopological spaces. Irresolute functions in bitopological spaces was defined by Mukherjee [35]. Khedr and Noiri introduced and studied in [21], [22] the notions of quasi-irresolute functions and almost s -continuous functions which are generalizations of weakly continuous functions between topological spaces due to Levine [23].

In [47]-[50], the present authors introduced and investigated the notions of minimal structures, m -spaces, m -continuous functions, M -continuous functions, weakly m -continuous functions and weakly M -continuous functions. Recently, in [39], [41] and other papers the present authors

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reduced the study of some forms of continuity between bitological spaces to the study of m -continuity and M -continuity between m -spaces.

Also Min and Kim [29]-[34] introduced and studied the notions of m -semi-open, m -preopen, α - m -open, β - m -open sets and M -semi-continuity, m -semi-continuity, $m\alpha$ -continuity. Quite recently, the notions of weakly M -semi-continuous functions and weakly M -precontinuous functions have been introduced in [33] and [34], respectively. And also these notions are introduced and studied in [8], [52] and other papers.

Quite recently, the first author [38] introduced the notion of bi- m -spaces which are called bi-minimal structure spaces in [5]. Some properties of biminimal structure spaces are studied in [4]-[6] and other papers. The purpose of this paper is to introduce and investigate the notion of weakly $M(i, j)$ -continuous functions. This function is a generalization of weak M -semicontinuity [33], weak M -precontinuity [34], (i, j) -weak m -continuity [43], (i, j) -weak quasi continuity [18], (i, j) -weak precontinuity [42], (i, j) -quasi irresoluteness [21] and (i, j) -almost s -continuity [22].

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X . The closure of A and the interior of A are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. We recall some generalized open sets in topological spaces.

DEFINITION 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

- (1) α -open [37] if $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$,
- (2) semi-open [24] if $A \subset \text{Cl}(\text{Int}(A))$,
- (3) preopen [27] if $A \subset \text{Int}(\text{Cl}(A))$,
- (4) β -open [1] or semi-preopen [3] if $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$.

The family of all α -open (resp. semi-open, preopen, β -open) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\text{SO}(X)$, $\text{PO}(X)$, $\beta(X)$).

DEFINITION 2.2. Let (X, τ) be a topological space. A subset A of X is said to be α -closed [28] (resp. semi-closed [9], preclosed [27], β -closed [1]) if the complement of A is α -open (resp. semi-open, preopen, β -open).

DEFINITION 2.3. Let (X, τ) be a topological space and A a subset of X . The intersection of all α -closed (resp. semi-closed, preclosed,

β -closed) sets of X containing A is called the α -closure [28] (resp. *semi-closure* [9], *preclosure* [10], β -closure [2]) of A and is denoted by $\alpha\text{Cl}(A)$ (resp. $\text{sCl}(A)$, $\text{pCl}(A)$, $\beta\text{Cl}(A)$).

DEFINITION 2.4. Let (X, τ) be a topological space and A a subset of X . The union of all α -open (resp. semi-open, preopen, β -open) sets of X contained in A is called the α -interior [28] (resp. *semi-interior* [9], *preinterior* [10], β -interior [2]) of A and is denoted by $\alpha\text{Int}(A)$ (resp. $\text{sInt}(A)$, $\text{pInt}(A)$, $\beta\text{Int}(A)$).

3. Minimal structures and bi- m -spaces

DEFINITION 3.1. Let X be a nonempty set and $P(X)$ the power set of X . A subfamily m_X of $P(X)$ is called a *minimal structure* (briefly *m-structure*) on X [47], [48] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X) , we denote a nonempty set X with an m -structure m_X on X and call it an m -space. Each member of m_X is said to be m_X -open (briefly m -open) and the complement of an m_X -open set is said to be m_X -closed (briefly m -closed).

REMARK 3.2. Let (X, τ) be a topological space. The families τ , $\alpha(X)$, $\text{SO}(X)$, $\text{PO}(X)$ and $\beta(X)$ are all minimal structures on X .

DEFINITION 3.3. Let X be a nonempty set and m_X an m -structure on X . For a subset A of X , the m_X -closure of A and the m_X -interior of A are defined in [26] as follows:

- (1) $\text{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,
- (2) $\text{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

REMARK 3.4. Let (X, τ) be a topological space and A a subset of X . If $m_X = \tau$ (resp. $\text{SO}(X)$, $\text{PO}(X)$, $\alpha(X)$, $\beta(X)$), then we have

- (1) $\text{mCl}(A) = \text{Cl}(A)$ (resp. $\text{sCl}(A)$, $\text{pCl}(A)$, $\alpha\text{Cl}(A)$, $\beta\text{Cl}(A)$),
- (2) $\text{mInt}(A) = \text{Int}(A)$ (resp. $\text{sInt}(A)$, $\text{pInt}(A)$, $\alpha\text{Int}(A)$, $\beta\text{Int}(A)$).

LEMMA 3.5. (Maki et al. [26]). Let X be a nonempty set and m_X a minimal structure on X . For subsets A and B of X , the following properties hold:

- (1) $\text{mCl}(X \setminus A) = X \setminus \text{mInt}(A)$ and $\text{mInt}(X \setminus A) = X \setminus \text{mCl}(A)$,
- (2) If $(X \setminus A) \in m_X$, then $\text{mCl}(A) = A$ and if $A \in m_X$, then $\text{mInt}(A) = A$,
- (3) $\text{mCl}(\emptyset) = \emptyset$, $\text{mCl}(X) = X$, $\text{mInt}(\emptyset) = \emptyset$ and $\text{mInt}(X) = X$,
- (4) If $A \subset B$, then $\text{mCl}(A) \subset \text{mCl}(B)$ and $\text{mInt}(A) \subset \text{mInt}(B)$,

- (5) $A \subset \text{mCl}(A)$ and $\text{mInt}(A) \subset A$,
- (6) $\text{mCl}(\text{mCl}(A)) = \text{mCl}(A)$ and $\text{mInt}(\text{mInt}(A)) = \text{mInt}(A)$.

LEMMA 3.6. (Popa and Noiri [47]). Let (X, m_X) be an m -space and A a subset of X . Then $x \in \text{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in m_X$ containing x .

DEFINITION 3.7. An m -structure m_X on a nonempty set X is said to have *property B* [26] if the union of any family of subsets belonging to m_X belongs to m_X .

REMARK 3.8. If (X, τ) is a topological space, then the m -structures $\text{SO}(X)$, $\text{PO}(X)$, $\alpha(X)$ and $\beta(X)$ have property B .

LEMMA 3.9. (Popa and Noiri [49]). Let X be a nonempty set and m_X an m -structure on X satisfying property B . For a subset A of X , the following properties hold:

- (1) $A \in m_X$ if and only if $\text{mInt}(A) = A$,
- (2) A is m_X -closed if and only if $\text{mCl}(A) = A$,
- (3) $\text{mInt}(A) \in m_X$ and $\text{mCl}(A)$ is m_X -closed.

DEFINITION 3.10. Let (X, m_X) be an m -space. A subset A of X is said to be

- (1) m - α -open [30] if $A \subset \text{mInt}(\text{mCl}(\text{mInt}(A)))$,
- (2) m -semi-open [29] if $A \subset \text{mCl}(\text{mInt}(A))$,
- (3) m -preopen [31] if $A \subset \text{mInt}(\text{mCl}(A))$,
- (4) m - β -open [6] if $A \subset \text{mCl}(\text{mInt}(\text{mCl}(A)))$.

The family of all m - α -open (resp. m -semi-open, m -preopen, m - β -open) sets in (X, m_X) is denoted by $\text{m}\alpha(X)$ (resp. $\text{mSO}(X)$, $\text{mPO}(X)$, $\text{m}\beta(X)$).

REMARK 3.11. Similar definitions of m -semi-open sets, m -preopen sets, m - α -open sets, m - β -open sets are provided in [8] and [52].

Let (X, m_X) be an m -space. We denote by $\text{mIT}(X)$ the family of all m -structures on X determined by iterating operators mInt and mCl ([44], [45]). However, in this paper, by $\text{mIT}(X)$ we denote $\text{m}\alpha(X)$, $\text{mSO}(X)$, $\text{mPO}(X)$ or $\text{m}\beta(X)$.

REMARK 3.12. (1) It easily follows from Lemma 3.5(3)-(4) that $\text{m}\alpha(X)$, $\text{mSO}(X)$, $\text{mPO}(X)$ and $\text{m}\beta(X)$ are minimal structures with property B . They are also shown in Theorem 3.5 of [29], Theorem 3.4 of [31] and Theorem 3.4 of [30].

- (2) Let (X, m_X) be an m -space and $\text{mIT}(X)$ an iterate structure on X . If $\text{mIT}(X) = \text{mSO}(X)$ (resp. $\text{mPO}(X)$, $\text{m}\alpha(X)$, $\text{m}\beta(X)$), then we obtain the following definitions provided in [29] (resp. [31], [30]):
 $\text{mITCl}(A) = \text{msCl}(A)$ (resp. $\text{mpCl}(A)$, $\text{m}\alpha\text{Cl}(A)$, $\text{m}\beta\text{Cl}(A)$),
 $\text{mITInt}(A) = \text{msInt}(A)$ (resp. $\text{mpInt}(A)$, $\text{m}\alpha\text{Int}(A)$, $\text{m}\beta\text{Int}(A)$).

In Theorem 4.2 of [50] and Theorems 7.4, 8.3 and 8.4 of [40], the authors used first m -spaces with two minimal structures. The first author [38] called a bi- m -space a nonempty set with two minimal structures on X . Recently, Boonpok [5] has renamed bi- m -spaces as biminimal structure spaces. In [4], the author studied some forms of continuity between two biminimal structure spaces.

Throughout the present paper, (X, τ_1, τ_2) (resp. (X, m_1, m_2)) denotes a bitopological space (resp. bi- m -space). Let (X, τ) be a topological space and A be a subset of X . Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X . The closure of A and the interior of A with respect to τ_i are denoted by $i\text{Cl}(A)$ and $i\text{Int}(A)$, respectively, for $i = 1, 2$. Similarly, we denote the m_X -closure of A and the m_X -interior of A with respect to m_i are denoted by $m_X^i\text{Cl}(A)$ and $m_X^i\text{Int}(A)$, respectively, for $i = 1, 2$.

REMARK 3.13. A bitopological space is a particular case of a bi- m -space.

Let (X, m_X) be an m -space and (Y, σ_1, σ_2) be a bitopological space. In [43], the authors introduced and studied a form of weakly continuous functions for a function $f : (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$.

DEFINITION 3.14. A function $f : (X, m_X) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) -weakly m -continuous [43] at $x \in X$ if for each $V \in \sigma_i$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset j\text{Cl}(V)$. The function f is said to be (i, j) -weakly m -continuous if it has this property at each point $x \in X$.

Recently, as weak forms of M -continuity [47], some functions between m -spaces are introduced and studied as follows:

DEFINITION 3.15. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be weakly M -continuous [49] $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset m\text{Cl}(V)$. The function f is said to be weakly M -continuous if it has this property at each point $x \in X$.

DEFINITION 3.16. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be weakly M -semicontinuous [33] at $x \in X$ if for each $V \in m_Y$ containing

$f(x)$, there exists $U \in \text{mSO}(X)$ containing x such that $f(U) \subset \text{msCl}(V)$. The function f is said to be *weakly M -semicontinuous* if it has this property at each point $x \in X$.

DEFINITION 3.17. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be *weakly M -precontinuous* [34] at $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in \text{mPO}(X)$ containing x such that $f(U) \subset \text{mpCl}(V)$. The function f is said to be *weakly M -precontinuous* if the function f has this property at each point $x \in X$.

Now we introduce a new function which is a generalization of the above four functions.

DEFINITION 3.18. A function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$ is said to be *weakly $M(i, j)$ -continuous* at $x \in X$ if for each $V \in m_Y^i$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset m_Y^j \text{Cl}(V)$. The function f is said to be *weakly $M(i, j)$ -continuous* if the function f has this property at each point $x \in X$.

REMARK 3.19. (1) If we set $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$ in Definition 3.18, then we obtain the definition of (i, j) -weakly m -continuity (Definition 3.14).

- (2) If we set $m_Y = \sigma_1 = \sigma_2$ in Definition 3.18, then we obtain the definition of weakly M -continuity (Definition 3.6).
- (3) If we set $m_X = \text{mSO}(X)$, $m_Y^1 = m_Y$ and $m_Y^2 = \text{mSO}(Y)$ in Definition 3.18, then a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is weakly M -semicontinuous if and only if $f : (X, \text{mSO}(X)) \rightarrow (Y, m_Y, \text{mSO}(Y))$ is weakly $M(1, 2)$ -continuous.
- (4) If we set $m_X = \text{mPO}(X)$, $m_Y^1 = m_Y$ and $m_Y^2 = \text{mPO}(Y)$ in Definition 3.18, then a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is weakly M -precontinuous if and only if $f : (X, \text{mPO}(X)) \rightarrow (Y, m_Y, \text{mPO}(Y))$ is weakly $M(1, 2)$ -continuous.

4. Characterizations of weak $M(i, j)$ -continuity

THEOREM 4.1. For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, the following properties are equivalent:

- (1) f is weakly $M(i, j)$ -continuous at $x \in X$;
- (2) for every $V \in m_Y^i$ containing $f(x)$, $x \in m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$;
- (3) for every m_Y^i -closed set F of Y such that $x \in m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F)))$, $x \in f^{-1}(F)$.

Proof. (1) \Rightarrow (2): Let $V \in m_Y^i$ containing $f(x)$. Then, by (1) there exists $U \in m_X$ containing x such that $f(U) \subset m_Y^j \text{Cl}(V)$. Thus $x \in U \subset f^{-1}(m_Y^j \text{Cl}(V))$ and hence $x \in m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$.

(2) \Rightarrow (3): Let F be any m_Y^j -closed set of Y . Suppose that $x \notin f^{-1}(F)$. Then $Y \setminus F \in m_Y^j$ and $x \in X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$. By (2) and Lemma 3.5, $x \in m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(Y \setminus F))) = m_X \text{Int}(f^{-1}(Y \setminus m_Y^j \text{Int}(F))) = X \setminus m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F)))$. Hence $x \notin m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F)))$.

(3) \Rightarrow (1): Let V be any m_Y^i -open set containing $f(x)$. Then $x \notin f^{-1}(Y \setminus V)$ and $Y \setminus V$ is m_Y^j -closed. By (3), $x \notin m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(Y \setminus V))) = m_X \text{Cl}(f^{-1}(Y \setminus m_Y^j \text{Cl}(V))) = m_X \text{Cl}(X \setminus f^{-1}(m_Y^j \text{Cl}(V))) = X \setminus m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$. Therefore, there exists $U \in m_X$ containing x such that $U \subset f^{-1}(m_Y^j \text{Cl}(V))$; hence $f(U) \subset m_Y^j \text{Cl}(V)$. \square

COROLLARY 4.2. *For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, the following properties are equivalent:*

- (1) f is weakly M -semicontinuous at $x \in X$;
- (2) for every $V \in m_Y$ containing $f(x)$, $x \in m_X \text{Int}(f^{-1}(m_Y \text{Cl}(V)))$;
- (3) for every m_Y -closed set F of Y such that $x \in m_X \text{Cl}(f^{-1}(m_Y \text{Int}(F)))$, $x \in f^{-1}(F)$.

REMARK 4.3. By Remark 3.19, we can obtain a quite similar characterizations of weak M -precontinuity from Theorem 4.1.

DEFINITION 4.4. A subset B of a bi- m -space (Y, m_Y^1, m_Y^2) is said to be m_{ij} -regular closed [4] if $B = m_Y^i \text{Cl}(m_Y^j \text{Int}(B))$.

THEOREM 4.5. *For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, where m_Y^1 and m_Y^2 have property B , the following properties are equivalent:*

- (1) f is weakly $M(i, j)$ -continuous at $x \in X$;
- (2) for every subset B of Y with $x \in m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(B))))$, $x \in f^{-1}(m_Y^i \text{Cl}(B))$;
- (3) for every m_{ij} -regular closed set F of Y such that $x \in m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F)))$, $x \in f^{-1}(F)$;
- (4) for every m_Y^j -open set V of Y with $x \in m_X \text{Cl}(f^{-1}(V))$, $x \in f^{-1}(m_Y^i \text{Cl}(V))$.

Proof. (1) \Rightarrow (2): Let B be any subset of Y with $x \in m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(B))))$. Since m_Y^i has property B , by Lemma 3.9, $m_Y^i \text{Cl}(B)$ is m_Y^j -closed. Then, by Theorem 4.1,

$x \in m_X \text{Cl}(f^{-1}(m_Y^i \text{Int}(m_Y^j \text{Cl}(B))))$ implies $x \in f^{-1}(m_Y^i \text{Cl}(B))$.

(2) \Rightarrow (3): Let F be any m_{ij} -regular closed set of Y such that $x \in m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F)))$. By (2), $x \in f^{-1}(m_Y^i \text{Cl}(m_Y^j \text{Int}(B))) = f^{-1}(F)$.

(3) \Rightarrow (4): Let V be any m_Y^j -open set of Y with $x \in m_X \text{Cl}(f^{-1}(V))$. Then $m_Y^i \text{Cl}(V) = m_Y^i \text{Cl}(m_Y^j \text{Int}(V))$ and $m_Y^i \text{Cl}(V)$ is m_{ij} -regular closed. By assumption, $x \in m_X \text{Cl}(f^{-1}(V)) = m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(V))) \subset m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(V))))$. By (3), $x \in f^{-1}(m_Y^i \text{Cl}(V))$.

(4) \Rightarrow (1): Let V be any m_Y^i -open set of Y containing $f(x)$. Since m_Y^j has property B , $m_Y^j \text{Cl}(V)$ is m_Y^j -closed and $Y \setminus m_Y^j \text{Cl}(V)$ is m_Y^j -open. Suppose that $x \notin m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$. Then

$$\begin{aligned} x &\in X \setminus m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V))) = m_X \text{Cl}(X \setminus f^{-1}(m_Y^j \text{Cl}(V))) \\ &= m_X \text{Cl}(f^{-1}(Y \setminus m_Y^j \text{Cl}(V))). \end{aligned}$$
 Since $Y \setminus m_Y^j \text{Cl}(V)$ is m_Y^j -open, by (4) $x \in f^{-1}(m_Y^i \text{Cl}(Y \setminus m_Y^j \text{Cl}(V))) = f^{-1}(Y \setminus m_Y^i \text{Int}(m_Y^j \text{Cl}(V))) \subset f^{-1}(Y \setminus m_Y^i \text{Int}(V)) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Hence $x \notin f^{-1}(V)$ and $f(x) \notin V$. This is a contradiction. Therefore, we obtain that $x \in m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$. By Theorem 4.1, f is weakly $M(i, j)$ -continuous at x . \square

COROLLARY 4.6. *For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, where m_Y has property B , the following properties are equivalent:*

- (1) f is weakly M -semicontinuous at $x \in X$;
- (2) for every subset B of Y with $x \in m_X \text{Cl}(f^{-1}(m_X \text{Int}(m_Y \text{Cl}(B))))$, $x \in f^{-1}(m_Y \text{Cl}(B))$;
- (3) for every m_{ij} -regular closed set F of Y such that $x \in m_X \text{Cl}(f^{-1}(m_X \text{Int}(F)))$, $x \in f^{-1}(F)$;
- (4) for every m_X -open set V of Y with $x \in m_X \text{Cl}(f^{-1}(V))$, $x \in f^{-1}(m_Y \text{Cl}(V))$.

REMARK 4.7. By Remark 3.19, we can obtain a quite similar characterizations of weak M -precontinuity from Theorem 4.5.

For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, we define $D_{M(i,j)}(f)$ as follows:

$$D_{M(i,j)}(f) = \{x \in X : f \text{ is not weakly } M(i, j)\text{-continuous at } x\}.$$

THEOREM 4.8. *For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, the following properties hold:*

$$\begin{aligned} D_{M(i,j)}(f) &= \bigcup_{G \in m_Y^i} \{f^{-1}(G) \setminus m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(G)))\} \\ &= \bigcup_{F \in \mathcal{F}} \{m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F))) \setminus f^{-1}(F)\}, \end{aligned}$$

where \mathcal{F} is the family of m_Y^j -closed sets of Y .

Proof. We show only the first equality because the proof of the other is similar to the first one. Let $x \in D_{M(i,j)}(f)$. By Theorem 4.1, there exists $V \in m_Y^i$ such that $f(x) \in V$ and $x \notin m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$. Therefore, we have $x \in f^{-1}(V) \setminus m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V))) \subset \bigcup_{G \in m_Y^i} \{f^{-1}(G) \setminus m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(G)))\}$. Conversely, let $x \in \bigcup_{G \in m_Y^i} \{f^{-1}(G) \setminus m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(G)))\}$. There exists $V \in m_Y^i$ such that $x \in f^{-1}(V) \setminus m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$. By Theorem 4.1, $x \in D_{M(i,j)}(f)$. \square

For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, we define $D_{Ms}(f)$ as follows:

$$D_{Ms}(f) = \{x \in X : f \text{ is not weakly } M\text{-semicontinuous at } x\}.$$

Then by Remark 3.19 and Theorem 4.8 we obtain the following corollary.

COROLLARY 4.9. *For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, the following properties hold:*

$$\begin{aligned} D_{Ms}(f) &= \bigcup_{G \in m_Y} \{f^{-1}(G) \setminus \text{msInt}(f^{-1}(\text{msCl}(G)))\} \\ &= \bigcup_{F \in \mathcal{F}} \{\text{msCl}(f^{-1}(\text{msInt}(F))) \setminus f^{-1}(F)\}, \end{aligned}$$

where F is the family of m_Y -closed sets of Y .

REMARK 4.10. By Remark 3.19, we can obtain a quite similar results of weak M -precontinuity from Theorem 4.8.

THEOREM 4.11. *For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, where m_Y^1 and m_Y^2 have property B , the following properties hold:*

$$\begin{aligned} D_{M(i,j)}(f) &= \bigcup_{B \in P(Y)} \{m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(B)))) \setminus f^{-1}(m_Y^i \text{Cl}(B))\} \\ &= \bigcup_{G \in m_Y^i} \{m_X \text{Cl}(f^{-1}(G)) \setminus f^{-1}(m_Y^i \text{Cl}(G))\} \\ &= \bigcup_{F \in \mathcal{F}} \{m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F))) \setminus f^{-1}(F)\}, \end{aligned}$$

where F is the family of m_{ij} regular closed sets of Y .

Proof. The proof is similar to that of Theorem 4.8. \square

COROLLARY 4.12. *For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, where m_Y has property B , the following properties hold:*

$$\begin{aligned} D_{Ms}(f) &= \bigcup_{B \in P(Y)} \{\text{msCl}(f^{-1}(\text{msInt}(m_Y \text{Cl}(B)))) \setminus f^{-1}(m_Y \text{Cl}(B))\} \\ &= \bigcup_{G \in \text{mSO}(Y)} \{\text{msCl}(f^{-1}(G)) \setminus f^{-1}(m_Y \text{Cl}(G))\} \\ &= \bigcup_{F \in \mathcal{F}} \{\text{msCl}(f^{-1}(\text{msInt}(F))) \setminus f^{-1}(F)\}, \end{aligned}$$

where F is the family of m_{ij} regular closed sets of Y .

REMARK 4.13. By Remark 3.19, we can obtain a quite similar results of weak M -precontinuity from Theorem 4.11.

THEOREM 4.14. For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, the following properties are equivalent:

- (1) f is weakly $M(i, j)$ -continuous;
- (2) $f^{-1}(V) \subset m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$ for every m_Y^i -open set V of Y ;
- (3) $m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F))) \subset f^{-1}(F)$ for every m_Y^i -closed set F of Y .

Proof. (1) \Rightarrow (2): Let V be any m_Y^i -open set and $x \in f^{-1}(V)$. Then, by Theorem 4.1 $x \in m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$ and hence $f^{-1}(V) \subset m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$.

(2) \Rightarrow (3): Let F be any m_Y^i -closed set of Y . Suppose that $x \notin f^{-1}(F)$. Then $Y \setminus F \in m_Y^i$ is and $x \in X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$. By (2) and Lemma 3.5, $x \in m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(Y \setminus F))) = m_X \text{Int}(f^{-1}(Y \setminus m_Y^j \text{Int}(F))) = X \setminus m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F)))$. Therefore, $x \notin m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F)))$ and hence $m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F))) \subset f^{-1}(F)$.

(3) \Rightarrow (1): Let V be any m_Y^i -open set and $x \in f^{-1}(V)$. Then $x \notin f^{-1}(Y \setminus V)$ and $Y \setminus V$ is m_Y^i -closed. By (3), $x \notin m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(Y \setminus V))) = m_X \text{Cl}(f^{-1}(Y \setminus m_Y^j \text{Cl}(V))) = m_X \text{Cl}(X \setminus f^{-1}(m_Y^j \text{Cl}(V))) = X \setminus m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$. Therefore, $x \in m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$. By Theorem 4.1, f is weakly $M(i, j)$ -continuous. \square

THEOREM 4.15. For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, where m_Y^1 and m_Y^2 have property B , the following properties are equivalent:

- (1) f is weakly $M(i, j)$ -continuous;
- (2) for every subset B of Y ,

$$m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(B)))) \subset f^{-1}(m_Y^i \text{Cl}(B));$$

- (3) for every m_{ij} -regular closed set F of Y ,

$$m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F))) \subset f^{-1}(F);$$

- (4) for every m_Y^j -open set V of Y , $m_X \text{Cl}(f^{-1}(V)) \subset f^{-1}(m_Y^i \text{Cl}(V))$.

Proof. (1) \Rightarrow (2): Let B be any subset of Y . Suppose $x \in m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(B))))$. By Theorem 4.5, $x \in f^{-1}(m_Y^i \text{Cl}(B))$. Hence $m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(B)))) \subset f^{-1}(m_Y^i \text{Cl}(B))$.

(2) \Rightarrow (3): Let F be any m_{ij} -regular closed set of Y . By (2), $m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(F))) = m_X(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(m_Y^j \text{Int}(F))))) \subset f^{-1}(m_Y^i \text{Cl}(m_Y^j \text{Int}(F))) = f^{-1}(F)$.

(3) \Rightarrow (4): Let V be any m_Y^j -open set of Y . Then $m_Y^i \text{Cl}(V) = m_Y^i \text{Cl}(m_Y^j \text{Int}(V))$ and $m_Y^i \text{Cl}(V)$ is m_{ij} -regular closed. By (3),

$$m_X \text{Cl}(f^{-1}(V)) = m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(V))) \subset m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_Y^i \text{Cl}(V)))) \\ \subset f^{-1}(m_Y^i \text{Cl}(V)).$$

(4) \Rightarrow (1): Let V be any m_Y^i -open set of Y . Since m_Y^j has property B , $m_Y^j \text{Cl}(V)$ is m_Y^j -closed and $Y \setminus m_Y^j \text{Cl}(V)$ is m_Y^j -open. By (4), $m_X \text{Cl}(f^{-1}(Y \setminus m_Y^j \text{Cl}(V))) \subset f^{-1}(m_Y^i \text{Cl}(Y \setminus m_Y^j \text{Cl}(V))) \subset X \setminus f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) \subset m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$ and by Theorem 4.14 f is weakly $M(i, j)$ -continuous. \square

REMARK 4.16. (1) By Theorems 4.14 and 4.15, we obtain the results from Theorems 3.3, 3.5 and 3.6 of [33].

(2) By Theorems 4.14 and 4.15, we obtain the results from Theorems 3.3 and 3.5 of [34].

(3) Let $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorems 4.14 and 4.15 we obtain Theorem 3.1 of [43].

DEFINITION 4.17. Let (X, m_X^1, m_X^2) be a bi- m -space and A a subset of X . A point x of X is called an m_{ij} - θ -adherent point of A if $A \cap m_X^j \text{Cl}(U) \neq \emptyset$ for every m_i -open set U containing x .

The set of all m_{ij} - θ -adherent points of A is called the m_{ij} - θ -closure of A and is denoted by $m_{ij} \text{Cl}_\theta(A)$. If $A = m_{ij} \text{Cl}_\theta(A)$, then A is said to be m_{ij} - θ -closed. A subset A of X is said to be m_{ij} - θ -open if $X \setminus A$ is m_{ij} - θ -closed.

LEMMA 4.18. Let (X, m_X^1, m_X^2) be a bi- m -space, where m_X^i has property B . Then $m_{ij} \text{Cl}_\theta(A)$ is m_X^i -closed for each subset A of X .

Proof. Let $x \in X \setminus m_{ij} \text{Cl}_\theta(A)$. Then $x \notin m_{ij} \text{Cl}_\theta(A)$. Hence there exists $U_x \in m_X^i$ containing x such that $m_X^j \text{Cl}(U_x) \cap A = \emptyset$. Then $U_x \cap A = \emptyset$ which implies that $U_x \cap m_{ij} \text{Cl}_\theta(A) = \emptyset$. Indeed, suppose that $U_x \cap m_{ij} \text{Cl}_\theta(A) \neq \emptyset$. Then, there exists $y \in U_x \cap m_{ij} \text{Cl}_\theta(A)$. Therefore, $y \in U_x$ and $y \in m_{ij} \text{Cl}_\theta(A)$. Therefore, we have $m_X^j \text{Cl}(U_x) \cap A \neq \emptyset$. This is a contradiction. Hence $x \in U_x \subset X \setminus m_{ij} \text{Cl}_\theta(A)$. Since m_X^i has property B , $X \setminus m_{ij} \text{Cl}_\theta(A) = \cup U_x \in m_X^i$. It follows that $m_{ij} \text{Cl}_\theta(A)$ is m_X^i -closed. \square

LEMMA 4.19. Let (X, m_X^1, m_X^2) be a bi- m -space. If U is an m_X^i -open set, then $m_{ji} \text{Cl}_\theta(U) = m_X^j \text{Cl}(U)$.

Proof. Suppose that $x \notin m_{ji} \text{Cl}_\theta(U)$. Then there exists $V \in m_X^j$ containing x such that $m_X^i \text{Cl}(V) \cap U = \emptyset$; hence $V \cap U = \emptyset$. Therefore, $V \cap m_X^j \text{Cl}(U) = \emptyset$ and hence $x \notin m_X^j \text{Cl}(U)$. Therefore, $m_{ji} \text{Cl}_\theta(U) \supset$

$m_X^j \text{Cl}(U)$. Conversely, suppose that $x \notin m_X^j \text{Cl}(U)$. Then there exists $V \in m_X^j$ containing x such that $V \cap U = \emptyset$. Since $U \in m_X^i$, $U \cap m_X^i \text{Cl}(V) = \emptyset$ and hence $x \notin m_{ji} \text{Cl}_\theta(U)$. Hence $m_{ji} \text{Cl}_\theta(U) \subset m_X^j \text{Cl}(U)$. Hence $m_{ji} \text{Cl}_\theta(U) = m_X^j \text{Cl}(U)$. \square

THEOREM 4.20. *For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, where m_Y^1 and m_Y^2 have property B, the following properties are equivalent:*

- (1) f is weakly $M(i, j)$ -continuous;
- (2) $f(m_X \text{Cl}(A)) \subset m_{ij} \text{Cl}_\theta(f(A))$ for every subset A of X ;
- (3) $m_X \text{Cl}(f^{-1}(B)) \subset f^{-1}(m_{ij} \text{Cl}_\theta(B))$ for every subset B of Y ;
- (4) $m_X \text{Cl}(f^{-1}(m_Y^i \text{Int}(m_{ij} \text{Cl}_\theta(B)))) \subset f^{-1}(m_{ij} \text{Cl}_\theta(B))$ for every subset B of Y .

Proof. (1) \Rightarrow (2): Suppose that f is weakly $M(i, j)$ -continuous. Let A be any subset of X , $x \in m_X \text{Cl}(A)$ and V be an m_Y^i -open set of Y containing $f(x)$. Then, there exists an m_X -open set U containing x such that $f(U) \subset m_Y^i \text{Cl}(V)$. Since $x \in m_X \text{Cl}(A)$, by Lemma 3.6 we obtain $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U) \cap f(A) \subset m_Y^i \text{Cl}(V) \cap f(A)$. Therefore, we obtain $f(x) \in m_{ij} \text{Cl}_\theta(f(A))$ and hence $f(m_X \text{Cl}(A)) \subset m_{ij} \text{Cl}_\theta(f(A))$.

(2) \Rightarrow (3): Let B be any subset of Y . Then $f(m_X \text{Cl}(f^{-1}(B))) \subset m_{ij} \text{Cl}_\theta(f(f^{-1}(B))) \subset m_{ij} \text{Cl}_\theta(B)$ and hence $m_X \text{Cl}(f^{-1}(B)) \subset f^{-1}(m_{ij} \text{Cl}_\theta(B))$.

(3) \Rightarrow (4): Let B be any subset of Y . Then, by Lemma 4.18 $m_{ij} \text{Cl}_\theta(B)$ is m_Y^i -closed in Y and by using Lemmas 4.18 and 4.19 we obtain

$$\begin{aligned} m_X \text{Cl}(f^{-1}(m_Y^i \text{Int}(m_{ij} \text{Cl}_\theta(B)))) &\subset f^{-1}(m_{ij} \text{Cl}_\theta(m_Y^i \text{Int}(m_{ij} \text{Cl}_\theta(B)))) \\ &= f^{-1}(m_Y^i \text{Cl}(m_Y^i \text{Int}(m_{ij} \text{Cl}_\theta(B)))) \subset f^{-1}(m_Y^i \text{Cl}(m_{ij} \text{Cl}_\theta(B))) \\ &= f^{-1}(m_{ij} \text{Cl}_\theta(B)). \end{aligned}$$

(4) \Rightarrow (1): Let V be any m_Y^j -open set of Y . Then by Lemma 4.19, $V \subset m_Y^j \text{Int}(m_Y^i \text{Cl}(V)) = m_Y^j \text{Int}(m_{ij} \text{Cl}_\theta(V))$ and we have $m_X \text{Cl}(f^{-1}(V)) \subset m_X \text{Cl}(f^{-1}(m_Y^j \text{Int}(m_{ij} \text{Cl}_\theta(V)))) \subset f^{-1}(m_{ij} \text{Cl}_\theta(V)) = f^{-1}(m_Y^i \text{Cl}(V))$.

Thus we obtain $m_X \text{Cl}(f^{-1}(V)) \subset f^{-1}(m_Y^i \text{Cl}(V))$. It follows from Theorem 4.15 that f is weakly $M(i, j)$ -continuous. \square

DEFINITION 4.21. Let (X, m_X) be an m -space and A be a subset of X . The m_X -frontier of A [43], $m_X \text{Fr}(A)$, is defined as follows: $m_X \text{Fr}(A) = m_X \text{Cl}(A) \cap m_X \text{Cl}(X \setminus A) = m_X \text{Cl}(A) \setminus m_X \text{Int}(A)$.

THEOREM 4.22. *Let (X, m_X) be an m -space and (Y, m_Y^1, m_Y^2) a bi- m -space. The set of all points x of X at which a function $f : (X, m_X) \rightarrow$*

(Y, m_Y^1, m_Y^2) is not weakly $M(i, j)$ -continuous is identical with the union of all m_X -frontiers of the inverse images of the m_Y^j -closure of m_Y^i -open sets of Y containing $f(x)$.

Proof. Let x be a point of X at which f is not weakly $M(i, j)$ -continuous. Then, there exists a m_Y^i -open set V of Y containing $f(x)$ such that $U \cap (X \setminus f^{-1}(m_Y^j \text{Cl}(V))) \neq \emptyset$ for every m_X -open set U of X containing x . By Lemma 3.6, $x \in m_X \text{Cl}(X \setminus f^{-1}(m_Y^j \text{Cl}(V)))$. Since $x \in f^{-1}(m_Y^j \text{Cl}(V))$, $x \in m_X \text{Cl}(f^{-1}(m_Y^j \text{Cl}(V)))$ and $x \in m_X \text{Fr}(f^{-1}(m_Y^j \text{Cl}(V)))$.

Conversely, if f is weakly $M(i, j)$ -continuous at x , then for each m_Y^i -open set V of Y containing $f(x)$, there exists an m_X -open set U containing x such that $f(U) \subset m_Y^j \text{Cl}(V)$ and hence $x \in U \subset f^{-1}(m_Y^j \text{Cl}(V))$. Therefore, we obtain that $x \in m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(V)))$ and hence $x \notin m_X \text{Fr}(f^{-1}(m_Y^j \text{Cl}(V)))$. \square

COROLLARY 4.23. *The set of all points x of X at which a function $f : (X, m_X) \rightarrow (Y, m_Y)$ is not weakly M -semicontinuous is identical with the union of all $mSO(X)$ -frontiers of the inverse images of the m_Y -semi-closure of m_Y -open sets of Y containing $f(x)$.*

5. Weak $M(i, j)$ -continuity and M -continuity

DEFINITION 5.1. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be M -continuous [47] at $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. The function f is said to be M -continuous if it has this property at each $x \in X$.

DEFINITION 5.2. A function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$ is said to be M - i -continuous if $f : (X, m_X) \rightarrow (Y, m_Y^i)$ is M -continuous.

LEMMA 5.3. *For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, the following properties are equivalent:*

- (1) f is M - i -continuous;
- (2) $f^{-1}(V) = m_X \text{Int}(f^{-1}(V))$ for every m_Y^i -open set V of Y ;
- (3) $f^{-1}(F) = m_X \text{Cl}(f^{-1}(F))$ for every m_Y^i -closed set F of Y .

Proof. The proof follows from Definition 5.2 and Theorem 3.1 of [47]. \square

DEFINITION 5.4. A bi- m -space (X, m_X^1, m_X^2) is said to be m_{ij} -regular if for each $x \in X$ and each m_X^i -open set U containing x , there exists an m_X^i -open set V such that $x \in V \subset m_X^j \text{Cl}(V) \subset U$.

LEMMA 5.5. *If a bi- m -space (X, m_X^1, m_X^2) is m_{ij} -regular, then $m_{ij}\text{Cl}_\theta(F) = F$ for every m_X^i -closed set F of X .*

Proof. Let F be any m_X^i -closed set of X and $x \in m_{ij}\text{Cl}_\theta(F)$, then $m_X^j\text{Cl}(U) \cap F \neq \emptyset$ for every m_X^i -open set U containing x . Since X is m_{ij} -regular, there exists an m_X^i -open set V such that $x \in V \subset m_X^j\text{Cl}(V) \subset U$. Since $x \in V \in m_X^i\text{Cl}(V) \cap F \neq \emptyset$. This implies that $U \cap F \neq \emptyset$ and hence $x \in m_X^i\text{Cl}(F)$. Then, we have $F \subset m_{ij}\text{Cl}_\theta(F) \subset m_X^i\text{Cl}(F) = F$. \square

THEOREM 5.6. *Let (Y, m_Y^1, m_Y^2) be an m_{ij} -regular bi- m -space, where m_Y^1 and m_Y^2 have property B. For a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$, the following properties are equivalent:*

- (1) *f is M - i -continuous;*
- (2) *$f^{-1}(m_{ij}\text{Cl}_\theta(B)) = m_X\text{Cl}(f^{-1}(m_{ij}\text{Cl}_\theta(B)))$ for every subset B of Y ;*
- (3) *f is weakly $M(i, j)$ -continuous;*
- (4) *$f^{-1}(F) = m_X\text{Cl}(f^{-1}(F))$ for every m_{ij} - θ -closed set F of Y ;*
- (5) *$f^{-1}(V) = m_X\text{Int}(f^{-1}(V))$ for every m_{ij} - θ -open set V of Y .*

Proof. (1) \Rightarrow (2): Let B be any subset of Y . By Lemma 4.18, $m_{ij}\text{Cl}_\theta(B)$ is m_Y^i -closed in Y . It follows from Lemma 5.5 that $f^{-1}(m_{ij}\text{Cl}_\theta(B)) = m_X\text{Cl}(f^{-1}(m_{ij}\text{Cl}_\theta(B)))$;

(2) \Rightarrow (3): Let B be any subset of Y . Then by (2) and Lemma 3.5 we have

$$m_X\text{Cl}(f^{-1}(B)) \subset m_X\text{Cl}(f^{-1}(m_{ij}\text{Cl}_\theta(B))) = f^{-1}(m_{ij}\text{Cl}_\theta(B)).$$

By Theorem 4.20, f is weakly $M(i, j)$ -continuous.

(3) \Rightarrow (4): Let F be any m_{ij} - θ -closed set of Y . Then by Theorem 4.20, $m_X\text{Cl}(f^{-1}(F)) \subset f^{-1}(m_{ij}\text{Cl}_\theta(F)) = f^{-1}(F)$. By Lemma 3.5, $f^{-1}(F) = m_X\text{Cl}(f^{-1}(F))$.

(4) \Rightarrow (5): Let V be any m_{ij} - θ -open set of Y . Then by (4), $X \setminus f^{-1}(V) = f^{-1}(Y \setminus V) = m_X\text{Cl}(f^{-1}(Y \setminus V)) = X \setminus m_X\text{Int}(f^{-1}(V))$. Hence $f^{-1}(V) = m_X\text{Int}(f^{-1}(V))$.

(5) \Rightarrow (1): Since (Y, m_Y^1, m_Y^2) is m_{ij} -regular, by Lemma 5.5 $m_{ij}\text{Cl}_\theta(B) = B$ for every m_Y^i -closed set B of Y and hence every m_Y^i -open set is m_{ij} - θ -open. Therefore, by Lemma 5.3, f is M - i -continuous. \square

6. Minimal structures in bitopological spaces

DEFINITION 6.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

- (1) (i, j) -semi-open [25] if $A \subset j\text{Cl}(i\text{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,
- (2) (i, j) -preopen [14] if $A \subset i\text{Int}(j\text{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$,
- (3) (i, j) - α -open [15] if $A \subset i\text{Int}(j\text{Cl}(i\text{Int}(A)))$, where $i \neq j$, $i, j = 1, 2$,
- (4) (i, j) -semi-preopen (briefly (i, j) -sp-open) [20] if there exists an (i, j) -preopen set U such that $U \subset A \subset j\text{Cl}(U)$, where $i \neq j$, $i, j = 1, 2$.

The family of all (i, j) -semi-open (resp. (i, j) -preopen, (i, j) - α -open, (i, j) -sp-open) sets of (X, τ_1, τ_2) is denoted by $(i, j)\text{SO}(X)$ (resp. $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\text{SPO}(X)$).

REMARK 6.2. Let (X, τ_1, τ_2) be a bitopological space. Then $(i, j)\text{SO}(X)$, $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$ and $(i, j)\text{SPO}(X)$ are all m -structures on X .

In the following, we denote by $m_{ij}(X)$ a minimal structure on X determined by τ_1 and τ_2 as in Definition 6.1. If $m_{ij}(X) = (i, j)\text{SO}(X)$ (resp. $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$, $(i, j)\text{SPO}(X)$), by Definition 3.3 for a subset A of X we have

$$\begin{aligned} m_{ij}\text{Cl}(A) &= (i, j)\text{sCl}(A) \text{ [25]} \\ \text{(resp. } (i, j)\text{pCl}(A) \text{ [20], } (i, j)\alpha\text{Cl}(A) \text{ [36], } (i, j)\text{spCl}(A) \text{ [20],} \\ m_{ij}\text{Int}(A) &= (i, j)\text{sInt}(A) \text{ (resp. } (i, j)\text{pInt}(A), (i, j)\alpha\text{Int}(A), (i, j)\text{spInt}(A)). \end{aligned}$$

REMARK 6.3. Let (X, τ_1, τ_2) be a bitopological space. Then the families $(i, j)\text{SO}(X)$, $(i, j)\text{PO}(X)$, $(i, j)\alpha(X)$ and $(i, j)\text{SPO}(X)$ are all m -structures on X satisfying property B by Theorem 2 of [25] (resp. Theorem 4.2 of [16] or Theorem 3.2 of [20], Theorem 5 of [36], Theorem 3.2 of [20]).

Let (X, τ_1, τ_2) be a bitopological space and A a subset of X . A point x of X is called an (i, j) -semi- θ -adherent point [21] of A , if $A \cap (j, i)\text{sCl}(U) \neq \emptyset$ for every (i, j) -semi-open set U containing x . The set of all (i, j) -semi- θ -adherent points of A is called the (i, j) -semi- θ -closure of A and is denoted by $(i, j)\text{sCl}_\theta(A)$. If $A = (i, j)\text{sCl}_\theta(A)$, then A is said to be (i, j) -semi- θ -closed. A subset A of X is said to be (i, j) -semi- θ -open if $X \setminus A$ is (i, j) -semi- θ -closed. A bitopological space (X, τ_1, τ_2) is said to be (i, j) -semi-regular [21] if for each (i, j) -semi-open set G and each $x \in G$, there exists an (i, j) -semi-open set U such that $x \in U \subset (j, i)\text{sCl}(U) \subset G$.

DEFINITION 6.4. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) -weakly semi-continuous [18] (resp. (i, j) -weakly precontinuous [42]) if for each $x \in X$ and each σ_i -open set V of Y containing $f(x)$, there exists an (i, j) -semi-open (resp. (i, j) -preopen) set U containing x such that $f(U) \subset j\text{Cl}(V)$.

Hence, a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -weakly semi-continuous (resp. (i, j) -weakly precontinuous) if and only if a function $f : (X, (i, j)\text{SO}(X)) \rightarrow (Y, \sigma_1, \sigma_2)$ (resp. $f : (X, (i, j)\text{PO}(X)) \rightarrow (Y, \sigma_1, \sigma_2)$) is weakly $M(i, j)$ -continuous.

DEFINITION 6.5. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) -quasi irresolute [21] (resp. (i, j) -almost s -continuous [22]) if for each $x \in X$ and each (i, j) -semi-open set V of Y containing $f(x)$, there exists $U \in (i, j)\text{SO}(X)$ (resp. $U \in \tau_i$) containing x such that $f(U) \subset (j, i)\text{sCl}(V)$.

Hence, a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -quasi irresolute (resp. (i, j) -almost s -continuous) if and only if a function $f : (X, (i, j)\text{SO}(X)) \rightarrow (Y, (i, j)\text{SO}(Y), (j, i)\text{SO}(Y))$ (resp. $f : (X, \tau_i) \rightarrow (Y, (i, j)\text{SO}(Y), (j, i)\text{SO}(Y))$) is weakly $M(i, j)$ -continuous.

Therefore, by the results of Sections 4 and 5, we can obtain the results established in [18], [21], [22], [42] and [43].

REMARK 6.6. Let (X, τ_1, τ_2) be a bitopological space. Then the families $(i, j)\text{SO}(X)$ and $(i, j)\text{PO}(X)$ have property B .

- (1) If we set $m_X = (i, j)\text{SO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorems 4.14 and 4.15 we obtain the results established in Theorems 2.1 and 2.2 in [18] and Theorems 3.1 and 3.2 in [46].
- (2) If we set $m_X = (i, j)\text{PO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorems 4.14 and 4.15 we obtain the results established in Theorems 3.1 and 3.2 of [42].
- (3) If we set $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorems 4.14 and 4.15 we obtain the results established in Theorem 3.1 of [43].
- (4) If we set $m_X = (i, j)\text{SO}(X)$, $m_Y^1 = (i, j)\text{SO}(Y)$ and $m_Y^2 = (j, i)\text{SO}(Y)$, then by Theorem 4.14 we obtain the results established in Proposition 15 (1), (4), (5) of [21] and Theorem 2.4 (1), (2) of [19].

REMARK 6.7. (1) If we set $m_X = (i, j)\text{SO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 5.6 we obtain the results established in Theorem 3.2 in [51].

- (2) If we set $m_X = (i, j)\text{PO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 5.6 we obtain the results established in Theorems 3.3 of [42].
- (3) If we set $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 5.6 we obtain the results established in Theorem 4.1 of [43].
- (4) If we set $m_X = m_{ij}\text{SO}(X)$, $m_Y^1 = (i, j)\text{SO}(Y)$ and $m_Y^2 = (j, i)\text{SO}(Y)$, then by Theorem 5.6 we obtain the results established in Theorem 2.7 of [19].

REMARK 6.8. (1) If we set $m_X = (i, j)\text{SO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 4.22 we obtain the results established in Theorem 4.3 in [51].

- (2) If we set $m_X = (i, j)\text{PO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 4.22 we obtain the results established in Theorem 4.3 of [42].

By Theorem 4.15, we can obtain the characterizations of (i, j) -almost s -continuous functions.

COROLLARY 6.9. *For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

- (1) f is (i, j) -almost s -continuous;
- (2) for every subset B of Y ,

$$i\text{Cl}(f^{-1}((j, i)\text{sInt}((i, j)\text{sCl}(B)))) \subset f^{-1}((i, j)\text{sCl}(B));$$

- (3) for every semi-regular closed set F of Y ,

$$i\text{Cl}(f^{-1}((j, i)\text{sInt}(F))) \subset f^{-1}(F);$$

- (4) for every (j, i) -semi-open set V of Y , $i\text{Cl}(f^{-1}(V)) \subset f^{-1}((i, j)\text{sCl}(V))$.

REMARK 6.10. (1) If we set $m_X = (i, j)\text{SO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 4.20 we obtain the results established in Theorem 3.3 in [46].

- (2) If we set $m_X = (i, j)\text{PO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 4.20 we obtain the results established in Theorems 3.2 of [42].
- (3) If we set $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 5.6 we obtain the results established in Theorem 4.20 of [43].
- (4) If we set $m_X = m_{ij}\text{SO}(X)$, $m_Y^1 = (i, j)\text{SO}(Y)$ and $m_Y^2 = (j, i)\text{SO}(Y)$, then by Theorem 4.20 we obtain the results established in Theorem 2.3 of [19].

- (5) If we set $m_X = \tau_i$, $m_Y^1 = (i, j)\text{SO}(Y)$ and $m_Y^2 = (j, i)\text{SO}(Y)$, then by Theorem 4.20 we obtain new characterizations of (i, j) -almost s -continuous functions.

COROLLARY 6.11. *For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

- (1) f is (i, j) -almost s -continuous;
- (2) for every subset A of X , $f(i\text{Cl}(A)) \subset (i, j)\text{sCl}_\theta(f(A))$;
- (3) for every subset B of Y , $i\text{Cl}(f^{-1}(B)) \subset f^{-1}((i, j)\text{sCl}_\theta(B))$;
- (4) for every subset B of Y , $i\text{Cl}(f^{-1}((j, i)\text{Int}((i, j)\text{sCl}_\theta(B)))) \subset f^{-1}((i, j)\text{sCl}_\theta(B))$.

For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, we define $D_{(i,j)s}(f)$ as follows:

$$D_{(i,j)s}(f) = \{x \in X : f \text{ is not } (i, j)\text{-weakly semi-continuous at } x\}.$$

Then, by Theorems 4.8 and 4.11 we obtain the following corollary.

COROLLARY 6.12. *For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties hold:*

$$\begin{aligned} D_{(i,j)s}(f) &= \bigcup_{G \in \sigma_i} \{f^{-1}(G) \setminus (i, j)\text{sInt}(f^{-1}(j\text{Cl}(G)))\} \\ &= \bigcup_{F \in \mathcal{F}} \{(i, j)\text{sCl}(f^{-1}(j\text{Int}(F))) \setminus f^{-1}(F)\}, \\ &= \bigcup_{B \in P(Y)} \{(i, j)\text{sCl}(f^{-1}(j\text{Int}(i\text{Cl}(B)))) \setminus f^{-1}(i\text{Cl}(B))\} \\ &= \bigcup_{R \in \mathcal{R}} \{(i, j)\text{sCl}(f^{-1}(j\text{Int}(R))) \setminus f^{-1}(R)\}, \\ &= \bigcup_{G \in \sigma_j} \{(i, j)\text{sCl}(f^{-1}(G)) \setminus f^{-1}(i\text{Cl}(G))\}, \end{aligned}$$

where \mathcal{F} is the family of σ_i -closed sets of Y and \mathcal{R} is the family of (i, j) regular closed sets of Y .

REMARK 6.13. If we set as follows:

- (1) $m_X = (i, j)\text{PO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$,
- (2) $m_X = (i, j)\text{SO}(X)$, $m_Y^1 = (i, j)\text{SO}(Y)$ and $m_Y^2 = (j, i)\text{SO}(Y)$,
- (3) $m_X = \tau_i$, $m_Y^1 = (i, j)\text{SO}(Y)$ and $m_Y^2 = (j, i)\text{SO}(Y)$,

then by Theorems 4.8 and 4.11 we obtain the similar corollaries with Corollary 6.12 concerning (1) (i, j) -weakly precontinuity, (2) (i, j) -quasi irresoluteness, and (3) (i, j) -almost s -continuity, respectively.

7. Some properties of weak $M(i, j)$ -continuity

DEFINITION 7.1. A bi- m -space (X, m_X^1, m_X^2) is said to be M_{ij} -Urysohn if for each distinct points x, y of X there exist an m_X^i -open set U and an m_X^j -open set V such that $x \in U$ and $y \in V$ and $m_X^j\text{Cl}(U) \cap m_X^i\text{Cl}(V) = \emptyset$ for $i \neq j, i, j = 1, 2$.

REMARK 7.2. If (X, τ_1, τ_2) is a bitopological space, then we obtain the definition of a pairwise Urysohn space [7].

DEFINITION 7.3. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to have a *strongly M -closed graph* [49] if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in m_X$ containing x and $V \in m_Y$ containing y such that $[U \times m_Y \text{Cl}(V)] \cap G(f) = \emptyset$.

LEMMA 7.4. (Popa and Noiri [49]) A function $f : (X, m_X) \rightarrow (Y, m_Y)$ has a strongly M -closed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in m_X$ containing x and $V \in m_Y$ containing y such that $f(U) \cap m_Y \text{Cl}(V) = \emptyset$.

DEFINITION 7.5. A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to have a *strongly M -semi-closed graph* [33] (resp. *strongly M -preclosed graph* [34]) if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m -semi-open (resp. m -preopen) set U containing x and $V \in m_Y$ containing y such that $[U \times m_Y \text{Cl}(V)] \cap G(f) = \emptyset$.

REMARK 7.6. If $m_X = \text{mSO}(X)$ (resp. $\text{mPO}(X)$) in Definition 7.3, then we obtain Definition 7.5.

DEFINITION 7.7. A function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$ is said to have an *M_{ij} -strongly closed graph* if $f : (X, m_X) \rightarrow (Y, m_Y^i)$ has a strongly M -closed graph.

THEOREM 7.8. If a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$ is weakly $M(i, j)$ -continuous and Y is M_{ij} -Urysohn, then f has an M_{ij} -strongly closed graph.

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is M_{ij} -Urysohn, there exist an m_Y^i -open set U and an m_Y^j -open set V such that $f(x) \in U$ and $y \in V$, respectively, such that $m_Y^j \text{Cl}(U) \cap m_Y^i \text{Cl}(V) = \emptyset$. By weak $M(i, j)$ -continuity of f , there exists an m_X -open set G containing x such that $f(G) \subset m_Y^j \text{Cl}(U)$; hence $f(G) \cap m_Y^i \text{Cl}(V) = \emptyset$. By Lemma 7.4, f has an M_{ij} -strongly closed graph. \square

REMARK 7.9. If $m_X = \text{mSO}(X)$ (resp. $\text{mPO}(X)$), then we obtain Theorem 3.9 of [33] (resp. Theorem 3.9 of [34])

DEFINITION 7.10. An m -space (X, m_X) is said to be m - T_2 [47] if for each distinct points x, y of X there exist m_X -open sets U and V such that $x \in U$ and $y \in V$, respectively, such that $U \cap V = \emptyset$.

THEOREM 7.11. *If a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$ is a weakly $M(i, j)$ -continuous injection, where $m_Y^i \subset m_Y^j$, and f has an M_{ij} -strongly closed graph, then (X, m_X) is m - T_2 .*

Proof. Let $x_1 \neq x_2$. Then $f(x_1) \neq f(x_2)$ and $(x_1, f(x_2)) \notin G(f)$. Since $G(f)$ is M_{ij} -strongly closed, there exist $U \in m_X$ containing x_1 and $V \in m_Y^i$ containing $f(x_2)$ such that $f(U) \cap m_Y^j \text{Cl}(V) = \emptyset$. Since f is weakly $M(i, j)$ -continuous and $f(x_2) \in V \in m_Y^i$, there exists an m_X -open set W containing x_2 such that $f(W) \subset m_Y^j \text{Cl}(V)$. Since $m_Y^i \subset m_Y^j$, it follows that $m_Y^j \text{Cl}(V) \subset m_Y^i \text{Cl}(V)$. Hence $f(U) \cap f(W) = \emptyset$. This implies that $U \cap W = \emptyset$ and hence (X, m_X) is m - T_2 . \square

REMARK 7.12. If $f : (X, m\text{SO}(X)) \rightarrow (Y, m_Y, m\text{SO}(Y))$ (resp. $f : (X, m\text{PO}(X)) \rightarrow (Y, m_Y, m\text{PO}(Y))$), then we obtain Theorem 3.10 of [33] (resp. Theorem 3.10 of [34]).

THEOREM 7.13. *If $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$ is a function such that*

- (1) (Y, m_Y^1, m_Y^2) is M_{ij} -Urysohn,
 - (2) $f(x_i) \neq f(x_j)$,
 - (3) f is weakly $M(i, j)$ -continuous at x_i and weakly $M(j, i)$ -continuous at x_j for distinct points $x_i, x_j \in X$,
- then (X, m_X) is m - T_2 .*

Proof. Let x_1 and x_2 be distinct points and $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since (Y, m_Y^1, m_Y^2) is M_{ij} -Urysohn, there exist an m_Y^i -open set V_i and an m_Y^j -open set V_j such that $y_i \in V_i$, $y_j \in V_j$ and $m_X^j \text{Cl}(V_i) \cap m_X^i \text{Cl}(V_j) = \emptyset$. Since f is weakly $M(i, j)$ -continuous at x_i and $f(x_i) \in V_i \in m_Y^i$, there exists $U_i \in m_X$ containing x_i such that $f(U_i) \subset m_Y^j \text{Cl}(V_i)$. Since f is weakly $M(j, i)$ -continuous at x_j and $f(x_j) \in V_j \in m_Y^j$, there exists $U_j \in m_X$ containing x_j such that $f(U_j) \subset m_Y^i \text{Cl}(V_j)$. Hence $f(U_i) \cap f(U_j) = \emptyset$ which implies $U_i \cap U_j = \emptyset$. Therefore, (X, m_X) is m - T_2 . \square

DEFINITION 7.14. An m -space (X, m_X) is said to be *m -connected* [47], [48] if it is not expressed as the union of two disjoint nonempty m -open sets of X .

DEFINITION 7.15. A bi- m -space (X, m_X^1, m_X^2) is said to be *pairwise m -connected* if it cannot be expressed as the union of two nonempty disjoint sets $U \in m_X^1$ and $V \in m_X^2$.

THEOREM 7.16. *Let a function $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$ be a function, where m_X, m_Y^1 and m_Y^2 have property B . If f is a weakly $M(i, j)$ -continuous and weakly $M(j, i)$ -continuous surjection and (X, m_X) is m -connected, then (Y, m_Y^1, m_Y^2) is pairwise m -connected.*

Proof. Suppose that (Y, m_Y^1, m_Y^2) is not pairwise m -connected. Then, there exist a σ_Y^i -open set U and a σ_Y^j -open set V such that $U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset$ and $U \cup V = Y$. Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Moreover $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. Since f is weakly $M(i, j)$ -continuous and weakly $M(j, i)$ -continuous, by Theorem 4.14 we have $f^{-1}(U) \subset m_X \text{Int}(f^{-1}(m_Y^j \text{Cl}(U)))$ and $f^{-1}(V) \subset m_X \text{Int}(f^{-1}(m_Y^i \text{Cl}(V)))$. Since $V = X \setminus U \in m_Y^j$ and $U = X \setminus V \in m_Y^i$, by Lemma 3.9 $U = m_Y^j \text{Cl}(U)$ and $V = m_Y^i \text{Cl}(V)$. Therefore, we have $f^{-1}(U) \subset m_X \text{Int}(f^{-1}(U))$ and $f^{-1}(V) \subset m_X \text{Int}(f^{-1}(V))$. Hence by Lemma 3.5 $f^{-1}(U) = m_X \text{Int}(f^{-1}(U))$ and $f^{-1}(V) = m_X \text{Int}(f^{-1}(V))$. By Lemma 3.3, $f^{-1}(U)$ and $f^{-1}(V)$ are m_X -open sets in (X, m_X) . This shows that (X, m_X) is not m -connected. \square

DEFINITION 7.17. A subset K of an m -space (X, m_X) is said to be *m -compact* if [47], [48] if every cover of K by m_X -open sets has a finite subcover.

DEFINITION 7.18. A subset K of a bi- m -space (Y, m_Y^1, m_Y^2) is said to be *M_{ij} -quasi H -closed relative to Y* if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of K by m_Y^i -open sets of Y , there exists a finite subset Δ_0 of Δ such that $K \subset \cup\{m_Y^j \text{Cl}(U_\alpha) : \alpha \in \Delta_0\}$.

THEOREM 7.19. *If $f : (X, m_X) \rightarrow (Y, m_Y^1, m_Y^2)$ is weakly $M(i, j)$ -continuous and K is an m -compact set in (X, m_X) , then $f(K)$ is M_{ij} -quasi H -closed relative to Y .*

Proof. Let K be an m -compact set of X and $\{V_\alpha : \alpha \in \Delta\}$ any cover of $f(K)$ by m_Y^i -open sets of Y . For each $x \in K$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is weakly $M(i, j)$ -continuous, there exists $U_x \in m_X$ containing x such that $f(U_x) \subset m_Y^j \text{Cl}(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of K by m_X -open sets. Since K is m -compact, there exist a finite number of points, say x_1, x_2, \dots, x_n in K such that $K \subset \cup\{U_{x_k} : x_k \in K, k = 1, 2, \dots, n\}$. Therefore, we obtain $f(K) \subset \cup\{f(U_{x_k}) : x_k \in K, k = 1, 2, \dots, n\} \subset \cup\{m_Y^j \text{Cl}(V_{\alpha(x_k)}) : x_k \in K, k = 1, 2, \dots, n\}$. This shows that $f(K)$ is M_{ij} -quasi H -closed relative to Y . \square

REMARK 7.20. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces.

- (1) If we set $m_X = (i, j)\text{PO}(X)$, $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 7.19 we obtain the results established in Theorem 6.3 of [42].
- (2) If we set $m_Y^1 = \sigma_1$ and $m_Y^2 = \sigma_2$, then by Theorem 7.19 we obtain the results established in Theorem 5.3 of [43].
- (3) If we set $m_X = (i, j)\text{SO}(X)$, $m_Y^1 = (i, j)\text{SO}(Y)$ and $m_Y^2 = (j, i)\text{SO}(Y)$, then by Theorem 7.19 we obtain the results established in Proposition 17 of [21].
- (4) If we set $m_X = \tau_i$, $m_Y^1 = (i, j)\text{SO}(Y)$ and $m_Y^2 = (j, i)\text{SO}(Y)$, then by Theorem 7.19 we obtain the results established in Theorem 7 of [22].

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