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A POINT STAR-CONFIGURATION IN $\mathbb{P}^n$ HAVING GENERIC HILBERT FUNCTION

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Abstract. We find a necessary and sufficient condition for which a point star-configuration in $\mathbb{P}^n$ has generic Hilbert function. More precisely, a point star-configuration in $\mathbb{P}^n$ defined by general forms of degrees $d_1, \dots, d_s$ with $3 \le n \le s$ has generic Hilbert function if and only if $d_1 = \cdots = d_{s-1} = 1$ and $d_s = 1, 2$. Otherwise, the Hilbert function of a point star-configuration in $\mathbb{P}^n$ is NEVER generic.

1. Introduction

Throughout the paper, $R = \Bbbk[x_0, x_1, \dots, x_n]$ is an $(n+1)$-variable polynomial ring over an infinite field $\Bbbk$ of any characteristic, and the symbol $\mathbb{P}^n$ will denote the projective n-space over a field $\Bbbk$. Let $F_1, \dots, F_s$ be general forms in R of degrees $d_1 \le \cdots \le d_s$ with $2 \le n \le s$. The variety $\mathbb{X}$ in $\mathbb{P}^n$ of the ideal

$$\bigcap_{1 \le i_1 < \cdots < i_r \le s} (F_{i_1}, \dots, F_{i_r})$$

is called *a star-configuration in* $\mathbb{P}^n$ *of type* (r, s). If the F_i are all general linear forms in R, the star-configuration $\mathbb{X}$ is called *a linear star-configuration in* $\mathbb{P}^n$. In particular, if $r = n$, then the variety $\mathbb{X}$ is called *a point star-configuration in* $\mathbb{P}^n$, which we shall discuss in this paper.

Let I be a homogeneous ideal of R. Then the numerical function

$$\mathbf{H}_{R/I}(t) := \dim_k R_t - \dim_k I_t \ (t \ge 0)$$

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is called the *Hilbert function* of the ring R/I. If $I := I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote

$$\mathbf{H}_{\mathbb{X}}(t) := \mathbf{H}_{R/I_{\mathbb{X}}}(t) \quad \text{for } t \geq 0$$

and call it the *Hilbert function* of $\mathbb{X}$.

Many interesting problems in the study of Hilbert functions and minimal free resolutions of star-configurations in $\mathbb{P}^n$ have been extensively studied (see [1, 2, 3, 4, 5, 6]). In this paper, we study point star-configurations in $\mathbb{P}^n$ defined by general forms for $n \geq 3$ (see [4]). Based on the degrees of these general forms, we find a necessary and sufficient condition for a point star-configuration in $\mathbb{P}^n$ to have generic Hilbert function for $n \geq 3$ (see Theorem 2.8).

2. A star-configuration in $\mathbb{P}^n$

The authors [1] showed that a star-configuration $\mathbb{X}$ in $\mathbb{P}^2$ has generic Hilbert function if $\mathbb{X}$ is defined by general forms $F_1, \ldots, F_s$ of the same degree $d = 1$, or 2, and that $\mathbb{X}$ has NEVER generic Hilbert function if $\mathbb{X}$ is defined by general forms $F_1, \ldots, F_s$ of the same degree $d \geq 3$. In [6], the author generalized the result as follows.

THEOREM 2.1 ([6, Theorem 2.7]). *Let $\mathbb{X}$ be a star-configuration in $\mathbb{P}^2$ defined by general forms $F_1, \ldots, F_s$ of degrees $d_1, \ldots, d_s$ with $3 \leq s$. Then $\mathbb{X}$ has generic Hilbert function if and only if $d_i \leq 2$ for every $i = 1, \ldots, s$.*

The following is an immediate corollary of Theorem 3.4 in [4] (see also [2, 3]).

COROLLARY 2.2 ([4, Corollary 2.4]). *Let $\mathbb{X}$ be a linear star-configuration in $\mathbb{P}^n$ of type (n, s) with $2 \leq n \leq s$. Then $\mathbb{X}$ has generic Hilbert function i.e.,*

$$\mathbf{H}_{\mathbb{X}}(i) = \min \left\{ \deg(\mathbb{X}), \binom{i+n}{n} \right\}$$

for every $i \geq 0$.

The following examples are motivation to this study: under what conditions given point star configurations in $\mathbb{P}^n$ to have generic Hilbert functions.

EXAMPLE 2.3.

(a) Consider a point star-configuration $\mathbb{X}$ in $\mathbb{P}^3$ defined by 5-general forms of degrees $1, 1, 1, 1, 2$. Then, by Theorem 3.4 in [4], $\mathbb{X}$ has generic Hilbert function

$$\mathbf{H}_{\mathbb{X}} \quad : \quad 1 \quad 4 \quad 10 \quad 16 \quad \rightarrow .$$

(b) However, if we consider a point star-configuration $\mathbb{Y}$ in $\mathbb{P}^3$ defined by 5-forms of degrees $1, 1, 1, 2, 2$, then the Hilbert function of $\mathbb{Y}$ is

$$\mathbf{H}_{\mathbb{Y}} \quad : \quad 1 \quad 4 \quad 10 \quad 19 \quad 25 \quad \rightarrow,$$

which is not generic.

As seen in Example 2.3 (a), there exists a non-linear point star-configuration in $\mathbb{P}^3$ having generic Hilbert function. This motivates the following proposition, which generalises Corollary 2.2.

Lemma 2.4. *Let $\mathbb{X}$ be a point star-configuration in $\mathbb{P}^n$ defined by general forms of degrees $1 \leq d_1 \leq \cdots \leq d_s$ with $2 \leq n \leq s$. If $d_1 = \cdots = d_{s-1} = 1$ and $d_s = 2$, then $\mathbb{X}$ has generic Hilbert function.*

Proof. By Theorem 2.1, if $n = 2$, then $\mathbb{X}$ has generic Hilbert function. We assume $n > 2$. If $s = n$, then $\mathbb{X}$ is a complete intersection of 2-points in $\mathbb{P}^n$, and hence the result is immediate. Now suppose $n < s$.

Note that, by Theorem 3.4 in [4], the degrees of the minimal generators are $s - n + 1$ or $s - n + 2$. Hence the Hilbert function of $\mathbb{X}$ up to degrees $\leq s - n$ is

$$\mathbf{H}_{\mathbb{X}} \quad : \quad 1 \quad \binom{1+n}{n} \quad \cdots \quad \binom{(s-n)+n}{n} \quad \cdots ,$$

and

$$\deg(\mathbb{X}) = \binom{s-1}{n} + 2 \cdot \binom{s-1}{n-1}.$$

Recall that $\binom{\alpha}{\beta} = \binom{\alpha-1}{\beta} + \binom{\alpha-1}{\beta-1}$ for $1 \leq \beta \leq \alpha$. Since the ideal $I_{\mathbb{X}}$ has $\binom{s-1}{n-2}$-generators in degree $s - n + 1$, we get that

$$\begin{aligned} & \mathbf{H}_{\mathbb{X}}(s-n+1) = \binom{(s-n+1)+n}{n} - \binom{s-1}{n-2} = \binom{s+1}{n} - \binom{s-1}{n-2} \\ = \; & \binom{s-1}{n} + 2 \cdot \binom{s-1}{n-1} = \deg(\mathbb{X}), \end{aligned}$$

and thus $\mathbb{X}$ has generic Hilbert function, as we wished. □

Lemma 2.5. *Let $\mathbb{X}$ be a point star-configuration in $\mathbb{P}^n$ defined by general forms of degrees $1 \leq d_1 \leq \cdots \leq d_s$ with $3 \leq n \leq s$. If $d_1 = \cdots = d_{s-1} = 1$ and $2 < d_s$, then the Hilbert function of $\mathbb{X}$ is NEVER generic.*

Proof. If $s = n$, then $\mathbb{X}$ is a complete intersection in $\mathbb{P}^n$ and the first two values of the Hilbert function of $\mathbb{X}$ are 1 and 2. Moreover, since $\deg(\mathbb{X}) = d_s > 2$, the Hilbert function of $\mathbb{X}$ is not generic.

Now assume that $s > n$. Notice that $I_{\mathbb{X}}$ has $\binom{s-1}{n-2}$-generators in degree $s-n+1$, and $\deg(\mathbb{X}) = \binom{s-1}{n} + d_s \cdot \binom{s-1}{n-1}$. Moreover, since

$$\begin{array}{rl} & \deg(\mathbb{X}) - \mathbf{H}_{\mathbb{X}}(s-n+1) \\ = & \binom{s-1}{n} + d_s \cdot \binom{s-1}{n-1} - \left[\binom{(s-n+1)+n}{n} - \binom{s-1}{n-2}\right] \\ = & \binom{s-1}{n} + d_s \cdot \binom{s-1}{n-1} - \left[\binom{s+1}{n} - \binom{s-1}{n-2}\right] \\ = & \binom{s-1}{n} + d_s \cdot \binom{s-1}{n-1} - \left[\binom{s-1}{n} + 2\binom{s-1}{n-1} + \binom{s-1}{n-2} - \binom{s-1}{n-2}\right] \\ = & (d_s - 2) \cdot \binom{s-1}{n-1} > 0, \end{array}$$

which means that $\mathbb{X}$ does not have generic Hilbert function. This completes the proof. □

To show that the condition in Lemma 2.4 is actually an equivalent relation for a point star-configuration $\mathbb{X}$ in $\mathbb{P}^n$ to have generic Hilbert function, we need one more notion $\sigma(\mathbb{X})$ which is defined to be

$$\sigma(\mathbb{X}) = \min\{d \mid \mathbf{H}_{\mathbb{X}}(d-1) = \mathbf{H}_{\mathbb{X}}(d)\}.$$

We will use the following proposition, which generalizes Proposition 3.6 in [6], in the proof of the main theorem (see Theorem 2.8).

Proposition 2.6. *Let $\mathbb{X} := \mathbb{X}^{(n,s)}$ be a point star-configuration in $\mathbb{P}^n$ defined by general forms $F_1, \dots, F_s$ of degrees $1 \le d_1 \le \cdots \le d_s$ with $2 \le n \le s$. Then*

$$\sigma(\mathbb{X}) = \left[\sum_{i=1}^{s} d_i\right] - (n-1).$$

Proof. We shall prove this by double induction on n and s. If $n = 2$, then by Proposition 3.6 in [6] it holds. Suppose $n > 2$. If $s = n$, then $\mathbb{X}$ is a complete intersection, and thus the statement is true. Now we assume that $n < s$. By Proposition 2.6 in [3], for $t \ge 0$,

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(t) + \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(t-d_s) - \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(t-d_s)$$

Define $d := \sum_{i=1}^{s} d_i$, and $d' := \sum_{i=1}^{s-1} d_i = d - d_s$. By induction on s,

$$\sigma(\mathbb{X}^{(n,s-1)}) = d' - (n-1), \quad \text{i.e.,}$$

$$\mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d' - (n+1)) < \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d' - n) = \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d' - (n-1)).$$

Since $\mathbb{X}^{(n-1,s-1)}$ is arithmetically Cohen-Macaulay and $\sigma(\mathbb{X}^{(n-1,s-1)}) = d' - (n-2)$ as a point star-configuration in $\mathbb{P}^{n-1}$, we get that

$$
\begin{aligned}
& \mathbf{H}_{\mathbb{X}}(d-(n-1)) - \mathbf{H}_{\mathbb{X}}(d-n) \\
= & \big[\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-(n-1)) + \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d'-(n-1)) - \\
& \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-(n-1))\big] - \\
& \big[\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-n) + \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d'-n) - \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-n)\big] \\
= & \big[\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-(n-1)) - \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-n)\big] + \\
& \big[\mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d'-(n-1)) - \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d'-n)\big] - \\
& \big[\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-(n-1)) - \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-n)\big] \\
= & \big[\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-(n-1)) - \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-n)\big] - \\
& \big[\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-(n-1)) - \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-n)\big] \\
& \text{(since } \sigma(\mathbb{X}^{(n,s-1)}) = d'-(n-1) \text{ by induction on } s) \\
= & \Delta\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-(n-1)) - \Delta\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-(n-1)) \\
= & 0.
\end{aligned}
$$

By the same argument as above, we have that

$$
\begin{aligned}
& \mathbf{H}_{\mathbb{X}}(d-n) - \mathbf{H}_{\mathbb{X}}(d-(n+1)) \\
= & \Delta\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-n) - \Delta\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-n) > 0.
\end{aligned}
$$

This implies that

$$
\mathbf{H}_{\mathbb{X}}(d-(n+1)) < \mathbf{H}_{\mathbb{X}}(d-n) = \mathbf{H}_{\mathbb{X}}(d-(n-1)),
$$

and thus

$$
\sigma(\mathbb{X}) = d-(n-1) = \left[\textstyle\sum_{i=1}^{s} d_i\right] - (n-1),
$$

as we wished. □

Proposition 2.7. *Let $\mathbb{X}$ be a point star-configuration in $\mathbb{P}^n$ defined by general forms of degrees $1 \le d_1 \le \cdots \le d_s$ for $3 \le n \le s$. If $d_1 = \cdots = d_{\ell-1} = 1$ and $2 \le d_\ell \le \cdots \le d_s$ with $1 \le \ell \le s-1$, then the Hilbert function of $\mathbb{X}$ is NEVER generic.*

Proof. By Proposition 2.6 in [3],

$$
\begin{aligned}
& \deg(\mathbb{X}) - \mathbf{H}_{\mathbb{X}}(d-(n+1)) \\
= & \deg(\mathbb{X}) - \big[\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d-(n+1)) - \mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-(n+1)) \\
& \quad + \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d'-(n+1))\big] \\
= & \deg(\mathbb{X}) - \big[(d_s-1)\cdot\deg(\mathbb{X}^{(n-1,s-1)}) + \Delta\mathbf{H}_{\mathbb{X}^{(n-1,s-1)}}(d'-n) \\
& \quad + \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d'-(n+1))\big]
\end{aligned}
$$

$$
\begin{aligned}
> & \ \deg(\mathbb{X}) - \big[d_s \cdot \deg(\mathbb{X}^{(n-1,s-1)}) + \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d' - (n+1))\big] \\
& \qquad \text{(by Proposition 2.6, } \sigma(\mathbb{X}^{(n-1,s-1)}) = d' - (n-2) \text{ in } \mathbb{P}^{n-1}) \\
> & \ \big[\deg(\mathbb{X}) - d_s \cdot \deg(\mathbb{X}^{(n-1,s-1)})\big] - \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d' - (n+1)) \\
= & \ \deg(\mathbb{X}^{(n,s-1)}) - \mathbf{H}_{\mathbb{X}^{(n,s-1)}}(d' - (n+1)) \quad \text{(by Proposition 2.6 in [3])} \\
> & \ 0 \qquad \text{(by Proposition 2.6, } \sigma(\mathbb{X}^{(n,s-1)}) = d' - (n-1)).
\end{aligned}
$$

Notice that, by Theorem 3.4 in [4], $I_{\mathbb{X}}$ has at least one generator in degree $d - (d_{s-n+2} + \cdots + d_n) = d_1 + \cdots + d_{s-n+1}$. Moreover,

$$
\begin{aligned}
& [d - (n+1)] - [d_1 + \cdots + d_{s-n+1}] \\
= & \ (d_{s-n+2} + \cdots + d_{s-1} + d_s) - (n+1) \\
= & \ (d_{s-n+2} + \cdots + (d_{s-1} - 1) + (d_s - 1)) - (n-1) \\
\geq & \ 0 \qquad (\text{since } 2 \leq d_{s-1} \leq d_s),
\end{aligned}
$$

i.e., $d_1 + \cdots + d_{s-n+1} \leq d - (n+1)$, which implies that

$$\mathbf{H}_{\mathbb{X}}(d - (n+1)) < \binom{d-(n+1)+n}{n} = \binom{d-1}{n}.$$

Therefore, $\mathbf{H}_{\mathbb{X}}(d - (n+1)) < \min\{\deg(\mathbb{X}), \binom{d-1}{n}\}$, and thus the Hilbert function of $\mathbb{X}$ is not generic. This completes the proof. □

It is from Corollary 2.2, Lemmas 2.4, 2.5, and Proposition 2.7 that the following main theorem is immediate.

THEOREM 2.8. *Let $\mathbb{X} := \mathbb{X}^{(n,s)}$ be a point star-configuration in $\mathbb{P}^n$ defined by general forms of degrees $1 \leq d_1 \leq \cdots \leq d_s$ with $3 \leq n \leq s$. Then $\mathbb{X}$ has generic Hilbert function if and only if $d_1 = \cdots = d_{s-1} = 1$ and $d_s = 1, 2$.*

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