

OPPOSITE SKEW COPAIRED HOPF ALGEBRAS

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ABSTRACT. Let A be a Hopf algebra with a linear form $\sigma: k \rightarrow A \otimes A$, which is convolution invertible, such that $\sigma_{21}(\Delta \otimes id)\tau(\sigma(1)) = \sigma_{32}(id \otimes \Delta)\tau(\sigma(1))$. We define Hopf algebras, $(A_\sigma, m, u, \Delta_\sigma, \varepsilon, S_\sigma)$. If B and C are opposite skew copaired Hopf algebras and $A = B \otimes_k C$ then we find Hopf algebras, $(A_{[\sigma]}, m_B \otimes m_C, u_B \otimes u_C, \Delta_{[\sigma]}, \varepsilon_B \otimes \varepsilon_C, S_{[\sigma]})$. Let H be a finite dimensional commutative Hopf algebra with dual basis $\{h_i\}$ and $\{h_i^*\}$, and let $A = H^{op} \otimes H^*$. We show that if we define $\sigma: k \rightarrow H^{op} \otimes H^*$ by $\sigma(1) = \sum h_i \otimes h_i^*$ then $(A_{[\sigma]}, m_A, u_A, \Delta_{[\sigma]}, \varepsilon_A, S_{[\sigma]})$ is the dual space of Drinfeld double, $D(H)^*$, as Hopf algebra.

Let k be a field. All unadorned tensor products are over k and all maps are k -linear. For $f, g \in Hom(C, A)$, where C is a coalgebra and A is an algebra, $f * g$ is its convolution product $m_A(f \otimes g)\Delta_C$. Let $\tau: V \otimes W \rightarrow W \otimes V$ be the twist map given by $\tau(v \otimes w) = w \otimes v$. We use the sigma notation [4]; for $c \in C$, $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. If $(B, m_B, u_B, \Delta_B, \varepsilon_B, S_B)$ and $(C, m_C, u_C, \Delta_C, \varepsilon_C, S_C)$ are Hopf algebras then $(B \otimes_k C, m_{B \otimes C}, u_B \otimes u_C, \Delta_{B \otimes C}, \varepsilon_B \otimes \varepsilon_C, S_B \otimes S_C)$ has the Hopf algebra structure of the usual tensor product of algebras and the usual tensor product of coalgebras.

PROPOSITION 1. *If $(A, m, u, \Delta, \varepsilon, S)$ is a Hopf algebra with a convolution invertible linear form, $\sigma: k \rightarrow A \otimes A$, such that*

$$(*) \quad \begin{aligned} & \sum \sigma_{i2}(1)(\sigma_{j2}(1))_{(1)} \otimes \sigma_{i1}(1)(\sigma_{j2}(1))_{(2)} \otimes \sigma_{j1}(1) \\ &= \sum \sigma_{j2}(1) \otimes \sigma_{i2}(1)(\sigma_{j1}(1))_{(1)} \otimes \sigma_{i1}(1)(\sigma_{j1}(1))_{(2)}, \end{aligned}$$

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where $\sigma(1) = \sum \sigma_{i1}(1) \otimes \sigma_{i2}(1)$ and $\sigma^{-1}(1) = \sum \sigma_{j1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1)$,
then

- (1) $\sum \sigma_{i1}(1) \sigma_{j1}^{-1}(1) \otimes \sigma_{i2}(1) \sigma_{j2}^{-1}(1) = 1 \otimes 1,$
 $\sum \sigma_{j1}^{-1}(1) \sigma_{i1}(1) \otimes \sigma_{j2}^{-1}(1) \sigma_{i2}(1) = 1 \otimes 1.$
- (2) $\sum \varepsilon(\sigma_1(1)) \sigma_2(1) = 1, \quad \sum \sigma_1(1) \varepsilon(\sigma_2(1)) = 1.$
- (3) $\sum (\sigma_{i2}^{-1}(1))_{(1)} \sigma_{j2}^{-1}(1) \otimes (\sigma_{i2}^{-1}(1))_{(2)} \sigma_{j1}^{-1}(1) \otimes \sigma_{i1}^{-1}(1)$
 $= \sum \sigma_{i2}^{-1}(1) \otimes (\sigma_{i1}^{-1}(1))_{(1)} \sigma_{j2}^{-1}(1) \otimes (\sigma_{i1}^{-1}(1))_{(2)} \sigma_{j1}^{-1}(1).$
- (4) $\sum \varepsilon(\sigma_1^{-1}(1)) \sigma_2^{-1}(1) = 1, \quad \sum \sigma_1^{-1}(1) \varepsilon(\sigma_2^{-1}(1)) = 1.$
- (5) $\sum \sigma_{i1}(1) \varepsilon(\sigma_{j1}^{-1}(1)) \otimes \sigma_{j2}^{-1}(1) \varepsilon(\sigma_{i2}(1)) = 1 \otimes 1,$
 $\sum \sigma_{j1}^{-1}(1) \varepsilon(\sigma_{i1}(1)) \otimes \sigma_{i2}(1) \varepsilon(\sigma_{j2}^{-1}(1)) = 1 \otimes 1.$
- (6) $\sum S(\sigma_{i2}(1)) \sigma_{i1}(1) S^{-1}(\sigma_{j2}^{-1}(1)) \sigma_{j1}^{-1}(1) = 1,$
 $\sum S^{-1}(\sigma_{j2}^{-1}(1)) \sigma_{j1}^{-1}(1) S(\sigma_{i2}(1)) \sigma_{i1}(1) = 1.$

Proof. (1) : Since σ^{-1} is a convolution inverse of σ ,

$$1 \otimes 1 = u_{A \otimes A} \varepsilon_k(1) = (\sigma * \sigma^{-1})(1) = (m \circ (\sigma \otimes \sigma^{-1}) \circ \Delta)(1) = \sigma(1) \sigma^{-1}(1),$$

$$1 \otimes 1 = u_{A \otimes A} \varepsilon_k(1) = (\sigma^{-1} * \sigma)(1) = \sigma^{-1}(1) \sigma(1).$$

(2) : Applying $id \otimes \varepsilon \otimes id$ to both sides of (*), we have

$$\begin{aligned} \sum \sigma_{i2}(1) (\sigma_{j2}(1))_{(1)} \otimes \varepsilon(\sigma_{i1}(1)) \varepsilon((\sigma_{j2}(1))_{(2)}) \otimes \sigma_{j1}(1) \\ = \sum \sigma_{j2}(1) \otimes \varepsilon(\sigma_{i2}(1)) \varepsilon((\sigma_{j1}(1))_{(1)}) \otimes \sigma_{i1}(1) (\sigma_{j1}(1))_{(2)}. \end{aligned}$$

Therefore,

$$\sum \sigma_{i2}(1) \sigma_{j2}(1) \otimes \varepsilon(\sigma_{i1}(1)) \otimes \sigma_{j1}(1) = \sum \sigma_{j2}(1) \otimes \varepsilon(\sigma_{i2}(1)) \otimes \sigma_{i1}(1) \sigma_{j1}(1).$$

Multiplying $\sum \sigma_{k2}^{-1}(1) \otimes 1 \otimes \sigma_{k1}^{-1}(1)$ on the right,

$$\left(\sum \varepsilon(\sigma_{i1}(1)) \sigma_{i2}(1) \right) \otimes 1 \otimes 1 = 1 \otimes 1 \otimes \left(\sum \sigma_{i1}(1) \varepsilon(\sigma_{i2}(1)) \right),$$

by (1). Thus

$$\sum \sigma_{i1}(1) \varepsilon(\sigma_{i2}(1)) = 1 = \sum \varepsilon(\sigma_{i1}(1)) \sigma_{i2}(1).$$

(3): The condition $(*)$ can be expressed as

$$\sigma_{21}(\Delta \otimes id)\tau(\sigma(1)) = \sigma_{32}(id \otimes \Delta)\tau(\sigma(1)),$$

where $\sigma_{21} = \sum \sigma_2(1) \otimes \sigma_1(1) \otimes 1$ and $\sigma_{32} = \sum 1 \otimes \sigma_2(1) \otimes \sigma_1(1)$.

Since

$$1 = (\Delta \otimes id)(\tau(\sigma^{-1} * \sigma)(1)) = [(\Delta \otimes id)\tau(\sigma^{-1}(1))][(\Delta \otimes id)\tau(\sigma(1))]$$

and

$$[(\Delta \otimes id)\tau(\sigma^{-1}(1))][(\Delta \otimes id)\tau(\sigma(1))] = 1,$$

it follows that

$$(\Delta \otimes id)\tau(\sigma^{-1}(1)) = [(\Delta \otimes id)\tau(\sigma(1))]^{-1}.$$

Similarly,

$$(id \otimes \Delta)\tau(\sigma^{-1}(1)) = [(id \otimes \Delta)\tau(\sigma(1))]^{-1}.$$

Hence

$$\begin{aligned} (\Delta \otimes id)\tau(\sigma^{-1}(1)) \cdot \sigma_{21}^{-1} &= [(\Delta \otimes id)\tau(\sigma(1))]^{-1} \cdot \sigma_{21}^{-1} \\ &= [\sigma_{21} \cdot (\Delta \otimes id)\tau(\sigma(1))]^{-1} \\ &= [\sigma_{32} \cdot (id \otimes \Delta)\tau(\sigma(1))]^{-1} \\ &= [(id \otimes \Delta)\tau(\sigma(1))]^{-1} \cdot \sigma_{32}^{-1} \\ &= (id \otimes \Delta)\tau(\sigma^{-1}(1)) \cdot \sigma_{32}^{-1}, \end{aligned}$$

where third equality follows from $(*)$.

(4) : It is proved in a same way of the proof of (2), using (3).

(5) : From (2) and (4).

(6) : Using (1) and (5),

$$\begin{aligned}
& \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1) \\
&= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1)\varepsilon(\sigma_{l1}^{-1}(1)\sigma_{k1}(1))\varepsilon(\sigma_{l2}^{-1}(1) \\
&\quad \sigma_{k2}(1)) \\
&= \sum S(\sigma_{i2}(1)\varepsilon(\sigma_{l2}^{-1}(1)))\sigma_{i1}(1)\varepsilon(\sigma_{l1}^{-1}(1))S^{-1}(\sigma_{j2}^{-1}(1)\varepsilon(\sigma_{k2}(1)))\sigma_{j1}^{-1}(1) \\
&\quad \varepsilon(\sigma_{k1}(1)) \\
&= 1.
\end{aligned}$$

Similarly, $\sum S^{-1}(\sigma_2^{-1}(1))\sigma_1^{-1}(1)S(\sigma_2(1))\sigma_1(1) = 1$. $\square$

EXAMPLE 1. Every Hopf algebra satisfies the condition $(*)$ by setting $\sigma(1) = 1 \otimes 1$. Let kZ_2 be written multiplicatively as $\{1, g\}$, and assume $\text{char } k \neq 2$. Now let

$$\sigma(1) = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g).$$

Then one can easily check that σ satisfies the property $(*)$.

We will define a new comultiplication Δ_σ and a new antipode S_σ on a Hopf algebra $(A, m, u, \Delta, \varepsilon, S)$. Then the following shows that $(A, m, u, \Delta_\sigma, \varepsilon, S_\sigma)$ is a Hopf algebra.

THEOREM 1. Let $(A, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra with an invertible linear form, $\sigma : k \rightarrow A \otimes A$. Assume that σ satisfies the condition $(*)$. Define $A_\sigma = A$, as an algebra. If we define the coproduct Δ_σ and the antipode S_σ by

$$\begin{aligned}
\Delta_\sigma(a) &= \sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1), \\
S_\sigma(a) &= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S(a)S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1), \quad a \in A, \text{ then} \\
(A_\sigma, m, u, \Delta_\sigma, \varepsilon, S_\sigma) &\text{ is a Hopf algebra.}
\end{aligned}$$

Proof. By the property $(*)$ and Proposition 1, (3),

$$\begin{aligned}
& (\Delta_\sigma \otimes id)\Delta_\sigma(a) \\
&= (\Delta_\sigma \otimes id)(\sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1)) \\
&= \sum \Delta_\sigma(\sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1)) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1) \\
&= \sum \sigma_{k2}(1)(\sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1))_{(1)}\sigma_{l2}^{-1}(1) \otimes \sigma_{k1}(1)(\sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1))_{(2)}
\end{aligned}$$

$$\begin{aligned}
& \sigma_{l1}^{-1}(1) \otimes \sigma_{i1}(1) a_{(2)} \sigma_{j1}^{-1}(1) \\
&= \sum \sigma_{k2}(1) (\sigma_{i2}(1))_{(1)} (a_{(1)})_{(1)} (\sigma_{j2}^{-1}(1))_{(1)} \sigma_{l2}^{-1}(1) \otimes \sigma_{k1}(1) (\sigma_{i2}(1))_{(2)} \\
&\quad (a_{(1)})_{(2)} (\sigma_{j2}^{-1}(1))_{(2)} \sigma_{l1}^{-1}(1) \otimes \sigma_{i1}(1) a_{(2)} \sigma_{j1}^{-1}(1) \\
&= \sum \sigma_{i2}(1) a_{(1)} \sigma_{j2}^{-1}(1) \otimes \sigma_{k2}(1) (\sigma_{i1}(1))_{(1)} (a_{(2)})_{(1)} (\sigma_{j1}^{-1}(1))_{(1)} \sigma_{l2}^{-1}(1) \\
&\quad \otimes \sigma_{k1}(1) (\sigma_{i1}(1))_{(2)} (a_{(2)})_{(2)} (\sigma_{j1}^{-1}(1))_{(2)} \sigma_{l1}^{-1}(1) \\
&= \sum \sigma_{i2}(1) a_{(1)} \sigma_{j2}^{-1}(1) \otimes \sigma_{k2}(1) (\sigma_{i1}(1) a_{(2)} \sigma_{j1}^{-1}(1))_{(1)} \sigma_{l2}^{-1}(1) \\
&\quad \otimes \sigma_{k1}(1) (\sigma_{i1}(1) a_{(2)} \sigma_{j1}^{-1}(1))_{(2)} \sigma_{l1}^{-1}(1) \\
&= \sum \sigma_{i2}(1) a_{(1)} \sigma_{j2}^{-1}(1) \otimes \Delta_{\sigma}(\sigma_{i1}(1) a_{(2)} \sigma_{j1}^{-1}(1)) \\
&= (id \otimes \Delta_{\sigma})(\sum \sigma_{i2}(1) a_{(1)} \sigma_{j2}^{-1}(1) \otimes \sigma_{i1}(1) a_{(2)} \sigma_{j1}^{-1}(1)) \\
&= (id \otimes \Delta_{\sigma}) \Delta_{\sigma}(a) \quad a, b \in A.
\end{aligned}$$

Thus

$$(\Delta_{\sigma} \otimes id) \Delta_{\sigma} = (id \otimes \Delta_{\sigma}) \Delta_{\sigma}.$$

By Proposition 1, (2) and (4),

$$\begin{aligned}
& \sum \varepsilon(\sigma_{i2}(1) a_{(1)} \sigma_{j2}^{-1}(1)) \sigma_{i1}(1) a_{(2)} \sigma_{j1}^{-1}(1) \\
&= \sum \varepsilon(\sigma_{i2}(1)) \varepsilon(a_{(1)}) \varepsilon(\sigma_{j2}^{-1}(1)) \sigma_{i1}(1) a_{(2)} \sigma_{j1}^{-1}(1) = a, \quad a \in A.
\end{aligned}$$

Thus

$$(\varepsilon \otimes id) \Delta_{\sigma} = 1 \otimes .$$

Similarly,

$$(id \otimes \varepsilon) \Delta_{\sigma} = \otimes 1.$$

Hence $(A_{\sigma}, \Delta_{\sigma}, \varepsilon)$ is coassociative and counitary. By Proposition 1,

(1),

$$\begin{aligned}
& \Delta_\sigma(ab) \\
&= \sigma_{i2}(1)(ab)_{(1)}\sigma_{j2}^{-1}(1) \otimes \sigma_{i1}(1)(ab)_{(2)}\sigma_{j1}^{-1}(1) \\
&= \sigma_{i2}(1)a_{(1)}b_{(1)}\sigma_{j2}^{-1}(1) \otimes \sigma_{i1}(1)a_{(2)}b_{(2)}\sigma_{j1}^{-1}(1) \\
&= \sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1)\sigma_{k2}(1)b_{(1)}\sigma_{l2}^{-1}(1) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1)\sigma_{k1}(1) \\
&\quad b_{(2)}\sigma_{l1}^{-1}(1) \\
&= (\sum \sigma_{i2}(1)a_{(1)}\sigma_{j2}^{-1}(1) \otimes \sigma_{i1}(1)a_{(2)}\sigma_{j1}^{-1}(1))(\sum \sigma_{k2}(1)b_{(1)}\sigma_{l2}^{-1}(1) \\
&\quad \otimes \sigma_{k1}(1)b_{(2)}\sigma_{l1}^{-1}(1)) \\
&= \Delta_\sigma(a)\Delta_\sigma(b), a, b \in A.
\end{aligned}$$

Thus

$$\Delta_\sigma(ab) = \Delta_\sigma(a)\Delta_\sigma(b), \quad a, b \in A.$$

And $\Delta_\sigma(1) = \sigma_2(1)1_A\sigma_2^{-1}(1) \otimes \sigma_1(1)1_A\sigma_1^{-1}(1) = 1_A \otimes 1_A$.

Hence Δ_σ and ε are algebra homomorphisms, i.e., $(A_\sigma, m, u, \Delta_\sigma, \varepsilon)$ is a bialgebra. We can show that S_σ is an antipode of the bialgebra $(A_\sigma, m, u, \Delta_\sigma, \varepsilon)$:

$$\begin{aligned}
& (S_\sigma * id)(a) \\
&= m \circ (S_\sigma \otimes id)(\Delta_\sigma(a)) \\
&= \sum S_\sigma(\sigma_{k2}(1)a_{(1)}\sigma_{l2}^{-1}(1))(\sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)) \\
&= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S(\sigma_{k2}(1)a_{(1)}\sigma_{l2}^{-1}(1))S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1)\sigma_{k1}(1) \\
&\quad a_{(2)}\sigma_{l1}^{-1}(1) \\
&= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1) \\
&\quad \sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1) \\
&= \sum S(\sigma_{i2}(1))\sigma_{i1}(1)S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))S^{-1}(\sigma_{j2}^{-1}(1))\sigma_{j1}^{-1}(1) \\
&\quad \sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)\varepsilon(\sigma_{m1}(1)\sigma_{n1}^{-1}(1))\varepsilon(\sigma_{m2}(1)\sigma_{n2}^{-1}(1)) \\
&= \sum S(\sigma_{i2}(1))\varepsilon(\sigma_{n2}^{-1}(1))\sigma_{i1}(1)\varepsilon(\sigma_{n1}^{-1}(1))S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1)) \\
&\quad S^{-1}(\sigma_{j2}^{-1}(1))\varepsilon(\sigma_{m2}(1))\sigma_{j1}^{-1}(1)\varepsilon(\sigma_{m1}(1))\sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1) \\
&= \sum S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))\sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)
\end{aligned}$$

$$\begin{aligned}
 &= \sum S(\sigma_{l2}^{-1}(1))S(a_{(1)})S(\sigma_{k2}(1))\sigma_{k1}(1)a_{(2)}\sigma_{l1}^{-1}(1)\varepsilon(\sigma_{j1}^{-1}(1)\sigma_{i1}(1)) \\
 &\quad \varepsilon(\sigma_{j2}^{-1}(1)\sigma_{i2}(1)) \\
 &= \sum S(\sigma_{l2}^{-1}(1))\varepsilon(\sigma_{i2}(1))S(a_{(1)})S(\sigma_{k2}(1))\varepsilon(\sigma_{j2}^{-1}(1))\sigma_{k1}(1)\varepsilon(\sigma_{j1}^{-1}(1)) \\
 &\quad a_{(2)}\sigma_{l1}^{-1}(1)\varepsilon(\sigma_{i1}(1)) \\
 &= \sum S(a_{(1)})a_{(2)} \\
 &= \varepsilon(a)1_A \\
 &= (u \circ \varepsilon)(a), \quad a \in A, \text{ by Proposition 1, (5). Similarly,}
 \end{aligned}$$

$$\sum (id * S_\sigma) = u \circ \varepsilon.$$

Hence S_σ is an antipode of A_σ . Therefore, $(A_\sigma, m, u, \Delta_\sigma, \varepsilon, S_\sigma)$ is a Hopf algebra. $\square$

Let B and C be bialgebras. Recall the definition of skew copairing in [3], which is the dual concept of skew pairing in [5].

DEFINITION 1. Let B and C be bialgebras. We say that B and C are *skew copaired* if there exists a k -linear map $\sigma: k \rightarrow B \otimes C$, $\sigma(1) = \sum \sigma_1(1) \otimes \sigma_2(1)$ (called the *skew copairing*) such that the diagrams below commute :

$$\begin{array}{ccccc}
 k & \xrightarrow{\sigma} & B \otimes C & \xrightarrow{id \otimes \Delta_C} & B \otimes C \otimes C \\
 \Delta_k \downarrow & & & & \uparrow m_B \otimes id \otimes id \\
 k \otimes k & \xrightarrow{\sigma \otimes \sigma} & B \otimes C \otimes B \otimes C & \xrightarrow{id \otimes \tau \otimes id} & B \otimes B \otimes C \otimes C \\
 & & & & \\
 k & \xrightarrow{\sigma} & B \otimes C & \xrightarrow{\Delta_B \otimes id} & B \otimes B \otimes C \\
 \Delta_k \downarrow & & & & \uparrow id \otimes id \otimes m_C \\
 k \otimes k & \xrightarrow{\sigma \otimes \sigma} & B \otimes C \otimes B \otimes C & \xrightarrow{id \otimes \tau \otimes id} & B \otimes B \otimes C \otimes C \\
 & & k & \xrightarrow{id} & k & \xrightarrow{id} & k \\
 & & \downarrow 1_B \otimes & & \downarrow \sigma & & \downarrow \otimes 1_C \\
 & & B \otimes k & \xleftarrow{id \otimes \varepsilon_C} & B \otimes C & \xrightarrow{\varepsilon_B \otimes id} & k \otimes C
 \end{array}$$

The commutativity of the diagrams above can be expressed equationally in the following way:

$$\begin{aligned} \sum \sigma_1(1) \otimes (\sigma_2(1))_{(1)} \otimes (\sigma_2(1))_{(2)} &= \sum \sigma_{i1}(1) \sigma_{j1}(1) \otimes \sigma_{i2}(1) \otimes \sigma_{j2}(1) \cdots (1) \\ \sum (\sigma_1(1))_{(1)} \otimes (\sigma_1(1))_{(2)} \otimes \sigma_2(1) &= \sum \sigma_{i1}(1) \otimes \sigma_{j1}(1) \otimes \sigma_{i2}(1) \sigma_{j2}(1) \cdots (2) \\ \sum \varepsilon(\sigma_1(1)) \otimes \sigma_2(1) &= 1 \otimes 1, \quad \sum \sigma_1(1) \otimes \varepsilon(\sigma_2(1)) = 1 \otimes 1 \cdots (3) \end{aligned}$$

where $\sigma(1) = \sum \sigma_1(1) \otimes \sigma_2(1) \in B \otimes C$.

EXAMPLE 2. ([3]) Let H be a finite dimensional Hopf algebra. Define σ as the coevaluation map

$$\sigma : k \rightarrow H^{op} \otimes H^*, \quad \sigma(1) = \sum h_i \otimes h_i^*,$$

where $\{h_i\}, \{h_i^*\}$ are dual bases of H and H^* . Then we see that σ is a skew copairing on H^{op} and H^* .

Recall the definition of quasitriangular Hopf algebra in [1] and [2]. If B and C are skew copaired Hopf algebras with an invertible skew copairing then some properties are similar to those in quasitriangular Hopf algebras.

PROPOSITION 2. *If B and C are skew copaired Hopf algebras with an invertible skew copairing, $\sigma : k \rightarrow B \otimes C$, which is convolution invertible where $\sigma^{-1}(1) = \sum \sigma_1^{-1}(1) \otimes \sigma_2^{-1}(1)$, then*

- (1) $\sigma(1)\sigma^{-1}(1) = 1_B \otimes 1_C, \quad \sigma^{-1}(1)\sigma(1) = 1_B \otimes 1_C$
- (2) $\sum (\sigma_1^{-1}(1))_{(1)} \otimes (\sigma_1^{-1}(1))_{(2)} \otimes \sigma_2^{-1}(1)$
 $= \sum \sigma_{i1}^{-1}(1) \otimes \sigma_{j1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1) \sigma_{i2}^{-1}(1)$
- (3) $\sum \sigma_1^{-1}(1) \otimes (\sigma_2^{-1}(1))_{(1)} \otimes (\sigma_2^{-1}(1))_{(2)}$
 $= \sum \sigma_{i1}^{-1}(1) \sigma_{j1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1) \otimes \sigma_{i2}^{-1}(1)$
- (4) $\sum \varepsilon_B(\sigma_1^{-1}(1)) \sigma_2^{-1}(1) = 1, \quad \sum \sigma_1^{-1}(1) \varepsilon_C(\sigma_2^{-1}(1)) = 1.$

Proof. (1) : Since σ^{-1} is a convolution inverse of σ ,

$$\begin{aligned} 1 \otimes 1 &= u_{B \otimes C} \varepsilon_k(1) = (\sigma * \sigma^{-1})(1) \\ &= (m_{B \otimes C} \circ (\sigma \otimes \sigma^{-1}) \circ \Delta)(1) \\ &= \sigma(1) \sigma^{-1}(1). \end{aligned}$$

Similarly,

$$\sigma^{-1}(1) \sigma(1) = 1 \otimes 1.$$

(2) : Since

$$\begin{aligned} 1 &= (\Delta_B \otimes id)((\sigma * \sigma^{-1})(1)) \\ &= [(\Delta_B \otimes id)\sigma(1)][(\Delta_B \otimes id)\sigma^{-1}(1)], \end{aligned}$$

we have

$$[(\Delta_B \otimes id)\sigma(1)]^{-1} = (\Delta_B \otimes id)\sigma^{-1}(1).$$

By Definition 1, (2), $(\Delta_B \otimes id)\sigma(1) = \sigma_{13}\sigma_{23}$, where $\sigma_{13} = \sum \sigma_1(1) \otimes 1_B \otimes \sigma_2(1)$ and $\sigma_{23} = \sum 1_B \otimes \sigma_1(1) \otimes \sigma_2(1)$ in $B \otimes B \otimes C$, so $(\Delta_B \otimes id)\sigma^{-1}(1) = [(\Delta_B \otimes id)\sigma(1)]^{-1} = \sigma_{23}^{-1} \sigma_{13}^{-1}$.

Thus

$$\begin{aligned} (\Delta_B \otimes id)\sigma^{-1}(1) &= \sigma_{23}^{-1} \sigma_{13}^{-1} \\ &= \sum \sigma_{i1}^{-1}(1) \otimes \sigma_{j1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1) \sigma_{i2}^{-1}(1). \end{aligned}$$

(3) : Similarly,

$$\begin{aligned} (id \otimes \Delta_C)\sigma^{-1}(1) &= \sigma_{13}^{-1} \sigma_{12}^{-1} \\ &= \sum \sigma_{i1}^{-1}(1) \sigma_{j1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1) \otimes \sigma_{i2}^{-1}(1). \end{aligned}$$

(4) : Since

$$\begin{aligned} 1 \otimes 1 &= (id \otimes \varepsilon_C)((\sigma * \sigma^{-1})(1)) \\ &= (id \otimes \varepsilon_C)(\sigma(1) \sigma^{-1}(1)) \\ &= [(id \otimes \varepsilon_C)\sigma(1)][(id \otimes \varepsilon_C)\sigma^{-1}(1)], \end{aligned}$$

$1 \otimes 1 = [(id \otimes \varepsilon_C)\sigma(1)]^{-1} = (id \otimes \varepsilon_C)\sigma^{-1}(1) = \sum \sigma_1^{-1}(1) \otimes \varepsilon_C(\sigma_2^{-1}(1))$ by Definition 1,(3). Similarly, $\sum \varepsilon_B(\sigma_1^{-1}(1))\sigma_2^{-1}(1) = 1$. $\square$

LEMMA 1. *If B and C are skew copaired bialgebras with a skew copairing $\sigma: k \rightarrow B \otimes C$, then B^{op} and C^{cop} are skew copaired bialgebras with same skew copairing $\sigma: k \rightarrow B^{op} \otimes C^{cop}$.*

Proof. Since B and C are skew copaired bialgebras

$$\begin{aligned}
& ((id \otimes \Delta_C^{op})\sigma)(1) \\
&= (id \otimes \tau_C \circ \Delta_C)\sigma(1) \\
&= (id \otimes \tau_C)((id \otimes \Delta_C)\sigma(1)) \\
&= (id \otimes \tau_C)((m_B \otimes id \otimes id)(id \otimes \tau \otimes id)(\sigma \otimes \sigma)(1)) \\
&= (id \otimes \tau_C)(\sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{i2}(1) \otimes \sigma_{j2}(1)) \\
&= \sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{j2}(1) \otimes \sigma_{i2}(1) \\
&= (m_B^{op} \otimes id \otimes id)(id \otimes \tau \otimes id)(\sigma \otimes \sigma)(1),
\end{aligned}$$

and

$$((\Delta_B \otimes id)\sigma)(1) = (id \otimes id \otimes m_C)(id \otimes \tau \otimes id)(\sigma \otimes \sigma)(1)$$

by definition. $\square$

EXAMPLE 3. Let H be a finite dimensional Hopf algebra. As in Example 2, the coevaluation map $\sigma: k \rightarrow H^{op} \otimes H^*$, $\sigma(1) = \sum h_i \otimes h_i^*$ is a skew copairing on H^{op} and H^* . By Lemma 1, $\sigma: k \rightarrow (H^{op})^{op} \otimes (H^*)^{cop}$ is a skew copairing on H and $(H^*)^{cop}$.

PROPOSITION 3. *Let B and C be skew copaired Hopf algebras with bijective antipodes S_B, S_C respectively and with an invertible skew copairing $\sigma: k \rightarrow B \otimes C$. Then*

$$\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S_C(\sigma_2(1)), \quad \sigma(1) = \sum S_B^{-1}(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1).$$

Proof. By the Definition 1, (1) and (3),

$$\begin{aligned}
& \sigma(1)(id \otimes S_C)(\sigma(1)) \\
&= \sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{i2}(1)S_C(\sigma_{j2}(1)) \\
&= (id \otimes m)(id \otimes id \otimes S_C)(\sum \sigma_{i1}(1)\sigma_{j1}(1) \otimes \sigma_{i2}(1) \otimes \sigma_{j2}(1)) \\
&= (id \otimes m)(id \otimes id \otimes S_C)(\sum \sigma_1(1) \otimes (\sigma_2(1))_{(1)} \otimes (\sigma_2(1))_{(2)}) \\
&= (id \otimes m)(id \otimes id \otimes S_C)((id \otimes \Delta_C)(\sigma(1))) \\
&= (id \otimes \varepsilon)(\sigma(1)) \\
&= 1 \otimes 1.
\end{aligned}$$

Therefore

$$\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S_C(\sigma_2(1)).$$

Since S^{-1} is an anti-coalgebra morphism

$$\begin{aligned}
& \sigma^{-1}(1)(S_B^{-1} \otimes id)(\sigma^{-1}(1)) \\
&= \sum \sigma_{j1}^{-1}(1)S_B^{-1}(\sigma_{i1}^{-1}(1)) \otimes \sigma_{j2}^{-1}(1)\sigma_{i2}^{-1}(1) \\
&= (m_B \otimes id)(id \otimes S_B^{-1} \otimes id)(\sum \sigma_{j1}^{-1}(1) \otimes \sigma_{i1}^{-1}(1) \otimes \sigma_{j2}^{-1}(1)\sigma_{i2}^{-1}(1)) \\
&= (m_B \otimes id)(id \otimes S_B^{-1} \otimes id)(\sum (\sigma_1^{-1}(1))_{(2)} \otimes (\sigma_1^{-1}(1))_{(1)} \otimes \sigma_2^{-1}(1)) \\
&= \sum (\sigma_1^{-1}(1))_{(2)}S_B^{-1}((\sigma_1^{-1}(1))_{(1)}) \otimes \sigma_2^{-1}(1) \\
&= (\varepsilon \otimes id)(\sigma^{-1}(1)) \\
&= 1 \otimes 1,
\end{aligned}$$

by Proposition 2, (2) and (4). Therefore

$$\sigma(1) = \sum S_B^{-1}(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1),$$

as desired. □

COROLLARY 1. *Let B and C be skew copaired Hopf algebras with bijective antipodes S_B, S_C respectively and with an invertible skew copairing $\sigma: k \rightarrow B \otimes C$. Then*

$$\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S_C^{-1}(\sigma_2(1)).$$

Proof. If $(B, m_B, u_B, \Delta_B, \varepsilon_B, S_B)$ and $(C, m_C, u_C, \Delta_C, \varepsilon_C, S_C)$ are Hopf algebras then $(B^{op}, m_B^{op}, u_B, \Delta_B, \varepsilon_B, S_B^{-1})$ and $(C^{cop}, m_C, u_C, \Delta_C^{op}, \varepsilon_C, S_C^{-1})$ are Hopf algebras with antipode S_B^{-1} , and S_C^{-1} respectively [1]. By Lemma 1, B^{op} and C^{cop} are skew copaired Hopf algebras with skew copairing σ . By Proposition 3, $\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S_C^{-1}(\sigma_2(1))$. $\square$

EXAMPLE 4. Let $(H, m, u, \Delta, \varepsilon, S)$ be a finite dimensional Hopf algebra with basis $\{h_i\}$. Define σ as the coevaluation map $\sigma: k \rightarrow H^{op} \otimes H^*$, $\sigma(1) = \sum h_i \otimes h_i^*$ where $\{h_i^*\}$ is the dual basis of $\{h_i\}$. Then we see that σ is an invertible skew copairing on H^{op} and H^* . By Proposition 3 and Corollary 1, $\sigma^{-1}(1) = \sum h_i \otimes S^{-1}(h_i^*)$ and $\sigma(1) = \sum S^{-1}(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1)$. So $\sigma(1) = \sum S^{-1}(h_i) \otimes S^{-1}(h_i^*)$. Hence $\sigma^{-1}(1) = \sum \sigma_1(1) \otimes S(\sigma_2(1)) = \sum S^{-1}(h_i) \otimes S(S^{-1}(h_i^*)) = \sum S^{-1}(h_i) \otimes h_i^*$.

DEFINITION 2. Bialgebras B and C are *opposite skew copaired* if there exists a skew copairing $\sigma: k \rightarrow B \otimes C$ (is called the *opposite skew copairing on B and C*) such that $\sum \sigma_{i1}(1)\sigma_{j1}(1) = \sum \sigma_{j1}(1)\sigma_{i1}(1)$ where $\sigma(1) = \sum \sigma_{i1}(1) \otimes \sigma_{i2}(1) \in B \otimes C$.

The opposite skew copairity can be expressed equationally in the following way:

$$\begin{aligned} \sum \sigma_1(1) \otimes (\sigma_2(1))_{(1)} \otimes (\sigma_2(1))_{(2)} &= \sum \sigma_{j1}(1)\sigma_{i1}(1) \otimes \sigma_{i2}(1) \otimes \sigma_{j2}(1) \cdots (1) \\ \sum (\sigma_1(1))_{(1)} \otimes (\sigma_1(1))_{(2)} \otimes \sigma_2(1) &= \sum \sigma_{i1}(1) \otimes \sigma_{j1}(1) \otimes \sigma_{i2}(1)\sigma_{j2}(1) \cdots (2) \\ \sum \varepsilon(\sigma_1(1)) \otimes \sigma_2(1) &= 1 \otimes 1, \quad \sum \sigma_1(1) \otimes \varepsilon(\sigma_2(1)) = 1 \otimes 1 \cdots (3) \end{aligned}$$

where $\sigma(1) = \sum \sigma_{i1}(1) \otimes \sigma_{i2}(1) \in B \otimes C$.

EXAMPLE 5. Let H_4 be Sweedler's four-dimensional Hopf algebra over k , and assume $\text{char } k \neq 2$. As an algebra over k , H_4 is generated by g and x with relations

$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx.$$

The coalgebra structure and antipode are determine by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = (x \otimes g) + (1 \otimes x),$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g = g^{-1}, \quad S(x) = gx.$$

H_4 has a basis $\{1, g, x, gx\}$. Let kZ_2 be written multiplicatively as $\{1, a\}$, and assume $\text{char } k \neq 2$. Now let

$$\sigma(1) = \frac{1}{2}(1 \otimes 1 + 1 \otimes a + g \otimes 1 - g \otimes a) \in H_4 \otimes kZ_2.$$

Then one can easily check that σ is an invertible opposite skew copairing of (H_4, kZ_2) with $\sigma^{-1} = \sigma$. Therefore (H_4, kZ_2) are opposite skew copaired Hopf algebras.

PROPOSITION 4. *Let σ be an invertible opposite skew copairing on (B, C) and let $A = B \otimes_k C$, the usual tensor product of algebras. Then the linear form $[\sigma]$ on A defined by*

$$[\sigma]: k \mapsto (B \otimes C) \otimes (B \otimes C),$$

$1 \mapsto \sum [\sigma]_{(1)} \otimes [\sigma]_{(2)} = \sum (\sigma_1(1) \otimes 1) \otimes (1 \otimes \sigma_2(1))$ satisfies $(*)$ with inverse $[\sigma]^{-1}(1) = \sum (\sigma_1^{-1}(1) \otimes 1) \otimes (1 \otimes \sigma_2^{-1}(1))$.

Proof. By the Definition 2, (1) and (2),

$$\begin{aligned}
& \sum [\sigma]_{i2}(1)([\sigma]_{j2}(1))_{(1)} \otimes [\sigma]_{i1}(1)([\sigma]_{j2}(1))_{(2)} \otimes [\sigma]_{j1}(1) \\
&= \sum (1 \otimes \sigma_{i2}(1))(1 \otimes \sigma_{j2}(1))_{(1)} \otimes (\sigma_{i1}(1) \otimes 1)(1 \otimes \sigma_{j2}(1))_{(2)} \otimes (\sigma_{j1}(1) \otimes 1) \\
&= \sum (1 \otimes \sigma_{i2}(1))(1 \otimes (\sigma_{j2}(1))_{(1)}) \otimes (\sigma_{i1}(1) \otimes 1)(1 \otimes (\sigma_{j2}(1))_{(2)}) \otimes (\sigma_{j1}(1) \otimes 1) \\
&= \sum (1 \otimes \sigma_{i2}(1))(1 \otimes \sigma_{j2}(1)) \otimes (\sigma_{i1}(1) \otimes 1)(1 \otimes \sigma_{k2}(1)) \otimes (\sigma_{k1}(1) \sigma_{j1}(1) \otimes 1) \\
&= \sum (1 \otimes \sigma_{i2}(1) \sigma_{j2}(1)) \otimes (1 \otimes \sigma_{k2}(1))(\sigma_{i1}(1) \otimes 1) \otimes (\sigma_{k1}(1) \otimes 1)(\sigma_{j1}(1) \otimes 1) \\
&= \sum (1 \otimes \sigma_{j2}(1)) \otimes (1 \otimes \sigma_{k2}(1))((\sigma_{j1}(1))_{(1)} \otimes 1) \otimes (\sigma_{k1}(1) \otimes 1)((\sigma_{j1}(1))_{(2)} \otimes 1) \\
&= \sum (1 \otimes \sigma_{j2}(1)) \otimes (1 \otimes \sigma_{i2}(1))(\sigma_{j1}(1) \otimes 1)_{(1)} \otimes (\sigma_{i1}(1) \otimes 1)(\sigma_{j1}(1) \otimes 1)_{(2)} \\
&= \sum [\sigma]_{j2}(1) \otimes [\sigma]_{i2}(1)([\sigma]_{j1}(1))_{(1)} \otimes [\sigma]_{i1}(1)([\sigma]_{j1}(1))_{(2)},
\end{aligned}$$

as desired. $\square$

The following theorem shows that $(A_{[\sigma]}, m, u, \Delta_{[\sigma]}, \varepsilon, S_{[\sigma]})$ are new Hopf algebras.

THEOREM 2. *Let $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ and $(C, m_C, u_C, \Delta_C, \varepsilon_C)$ be opposite skew copaired bialgebras with an invertible opposite skew copairing σ . Let $[\sigma]$ be the linear form in Proposition 4. There exists a bialgebra structure on the vector space $B \otimes_k C$, such that $A_{[\sigma]} = B \otimes C$, the usual tensor product of algebras as algebra with unit $u_B \otimes u_C$ and its comultiplication is given by*

$$\Delta_{[\sigma]}(b \otimes c) = \sum (b_{(1)} \otimes \sigma_2(1)c_{(1)}\sigma_2^{-1}(1)) \otimes (\sigma_1(1)b_{(2)}\sigma_1^{-1}(1) \otimes c_{(2)}),$$

and its counit by

$$\varepsilon(b \otimes c) = \varepsilon_B(b)\varepsilon_C(c)$$

for all $b \in B, c \in C$.

If the bialgebras B and C have antipodes, respectively denote S_B and S_C , then the bialgebra $(A_{[\sigma]}, m_B \otimes m_C, u_B \otimes u_C, \Delta_{[\sigma]}, \varepsilon)$ is a Hopf

algebra with antipode $S_{[\sigma]}$ given by

$$S_{[\sigma]}(b \otimes c) = \sum \sigma_1(1)S_B(b)\sigma_1^{-1}(1) \otimes S_C(\sigma_2(1))S_C(c)S_C^{-1}(\sigma_2^{-1}(1)).$$

Proof. By Theorem 1, and by Proposition 4, $(A_{[\sigma]}, m, u, \Delta_{[\sigma]}, \varepsilon)$ is a bialgebra, if B and C are Hopf algebras, then $(A_{[\sigma]}, m, u, \Delta_{[\sigma]}, \varepsilon, S_{[\sigma]})$ is a Hopf algebra since $A_{[\sigma]} = B \otimes C$ is an algebra by the definition of A_σ . And if let $[\sigma]: k \rightarrow (B \otimes C) \otimes (B \otimes C)$, $1 \mapsto \sum [\sigma]_1(1) \otimes [\sigma]_2(1)$ then

$$\begin{aligned} \Delta_{[\sigma]}(b \otimes c) &= \sum [\sigma]_2(1)(b \otimes c)_{(1)}[\sigma]_2^{-1}(1) \otimes [\sigma]_1(1)(b \otimes c)_{(2)}[\sigma]_1^{-1}(1) \\ &= \sum (1 \otimes \sigma_2(1))(b_{(1)} \otimes c_{(1)})(1 \otimes \sigma_2^{-1}(1)) \otimes (\sigma_1(1) \otimes 1)(b_{(2)} \otimes c_{(2)})(\sigma_1^{-1}(1) \otimes 1) \\ &= \sum (b_{(1)} \otimes \sigma_2(1)c_{(1)}\sigma_2^{-1}(1)) \otimes (\sigma_1(1)b_{(2)}\sigma_1^{-1}(1) \otimes c_{(2)}), \end{aligned}$$

and

$$\begin{aligned} S_{[\sigma]}(b \otimes c) &= \sum S([\sigma]_2(1))[\sigma]_1(1)S(b \otimes c)S^{-1}([\sigma]_2^{-1}(1))[\sigma]_1^{-1}(1) \\ &= \sum S(1 \otimes \sigma_2(1))(\sigma_1(1) \otimes 1)(S(b) \otimes S(c))S^{-1}(1 \otimes \sigma_2^{-1}(1))(\sigma_1^{-1}(1) \otimes 1) \\ &= \sum (S_B(1) \otimes S_C(\sigma_2(1)))(\sigma_1(1) \otimes 1)(S_B(b) \otimes S_C(c))(S_B^{-1}(1) \otimes S_C^{-1}(\sigma_2^{-1}(1))) \\ &\quad (\sigma_1^{-1}(1) \otimes 1) \\ &= \sum \sigma_1(1)S_B(b)\sigma_1^{-1}(1) \otimes S_C(\sigma_2(1))S_C(c)S_C^{-1}(\sigma_2^{-1}(1)), \end{aligned}$$

as desired. $\square$

Recall the definition of coquasitriangular Hopf algebra in [2]. The dual space of Drinfeld double $D(H)^*$ is an interesting object for the study of coquasitriangular Hopf algebras. We are going to show that $(A_{[\sigma]}, m_A, u_A, \Delta_{[\sigma]}, \varepsilon_A, S_{[\sigma]})$ is the dual space of Drinfeld double, $D(H)^*$, as Hopf algebra. So we find a constuction of coquasitriangular Hopf algebras when H is a finite dimensional commutative Hopf algebra.

THEOREM 3. *Let $(H, m, u, \Delta, \varepsilon, S)$ be a finite dimensional commutative Hopf algebra with dual basis $\{h_i\}$ and $\{h_i^*\}$. Let $A = H^{op} \otimes H^*$. If we define $\sigma: k \rightarrow H^{op} \otimes H^*$ by $\sigma(1) = \sum h_i \otimes h_i^*$ then the Hopf algebra $(A_{[\sigma]}, m_A, u_A, \Delta_{[\sigma]}, \varepsilon_A, S_{[\sigma]})$ is the dual space of Drinfeld double, $D(H)^*$, as Hopf algebra.*

Proof. Since H is commutative, the skew copairing σ is an opposite skew copairing of bialgebras H^{op} and H^* with inverse $\sigma^{-1}(1) = \sum S^{-1}(h_i) \otimes h_i^*$ as in Example 4. As an algebra, $A = H^{op} \otimes H^*$ has the same product with $D(H)^*$, the usual tensor product of algebras. The comultiplication on $A_{[\sigma]}$ is

$$\begin{aligned} \Delta_{[\sigma]}(h \otimes f) &= \sum (h_{(1)} \otimes \sigma_2(1) f_{(1)} \sigma_2^{-1}(1)) \otimes (\sigma_1^{-1}(1) h_{(2)} \sigma_1(1) \otimes f_{(2)}) \\ &= \sum (h_{(1)} \otimes h_s^* f_{(1)} h_t^*) \otimes (S^{-1}(h_t) h_{(2)} h_s \otimes f_{(2)}) \end{aligned}$$

in $H^{op} \otimes H^*$ by Theorem 2. Therefore the bialgebra $A_{[\sigma]}$ is the dual space of Drinfeld double, $D(H)^*$, as bialgebra. By Proposition 3, and Corollary 1, $\sum h_t \otimes S(h_t^*) = \sigma^{-1}(1) = \sum h_i \otimes S^{-1}(h_i^*)$ and $\sigma(1) = \sum S^{-1}(\sigma_1^{-1}(1)) \otimes \sigma_2^{-1}(1)$. Hence $\sigma(1) = \sum S^{-1}(h_i) \otimes S^{-1}(h_i^*)$. The antipode on $A_{[\sigma]}$ is

$$\begin{aligned} S_{[\sigma]}(h \otimes f) &= \sum \sigma_1^{-1}(1) S(h) \sigma_1(1) \otimes S(\sigma_2(1)) S(f) S^{-1}(\sigma_2^{-1}(1)) \\ &= \sum h_t S(h) S^{-1}(h_i) \otimes S(S^{-1}(h_i^*)) S(f) S^{-1}(S(h_t^*)) \\ &= \sum h_t S(h) S^{-1}(h_i) \otimes h_i^* S(f) h_t^*, \end{aligned}$$

by Theorem 2. Therefore the Hopf algebra $A_{[\sigma]}$ is the dual space of Drinfeld double, $D(H)^*$, as Hopf algebra. $\square$

COROLLARY 2. Let $(H, m, u, \Delta, \varepsilon, S)$ be a finite dimensional commutative Hopf algebra with dual basis $\{h_i\}$ and $\{h_i^*\}$. Let $A = H^{op} \otimes H^*$. If we define $\sigma: k \rightarrow H^{op} \otimes H^*$ by $\sigma(1) = \sum h_i \otimes h_i^*$ then the Hopf algebra $(A_{[\sigma]}, m_A, u_A, \Delta_{[\sigma]}, \varepsilon_A, S_{[\sigma]})$ is coquasitriangular.

Proof. By [2, Proposition 10.3.14], the dual space of Drinfeld double, $D(H)^*$ is a coquasitriangular Hopf algebra with braiding $\langle | \rangle: D(H)^* \otimes D(H)^* \rightarrow k$ is given by $\langle h \otimes f | k \otimes g \rangle = \varepsilon(k)f(1) \langle g, h \rangle$, as desired. $\square$

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