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GENERALIZED HYERS-ULAM STABILITY OF CUBIC TYPE FUNCTIONAL EQUATIONS IN NORMED SPACES

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Abstract. In this paper, we solve the Hyers-Ulam stability problem for the following cubic type functional equation

$$\begin{aligned} & f(rx+sy)+f(rx-sy) \\ & = rs^2 f(x+y)+rs^2 f(x-y)+2r(r^2-s^2)f(x) \end{aligned}$$

in quasi-Banach space and non-Archimedean space, where $r \neq \pm 1, 0$ and s are real numbers.

1. Introduction

In [26], S.M. Ulam proposed the stability problem for functional equations concerning the stability of group homomorphisms. A functional equation is called stable if any approximate solution to the functional equation is near a true solution of that functional equation. In [11], D.H. Hyers considered the case of approximate additive mappings with the Cauchy difference controlled by a positive constant in Banach spaces. D.G. Bourgin [4] and T. Aoki [2] treated this problem for approximate additive mappings controlled by unbounded function. In [21], Th. M. Rassias provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded. In 1994, P. Găvruta [8] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions. During the last three decades a number of papers and research

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monographs have been published on various generalizations and applications of the Hyers–Ulam stability and generalized Hyers–Ulam stability to a number of functional equations and mappings [1, 5, 7, 13, 20].

A stability problem of Ulam for the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

was first proved by F. Skof for mapping $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space [24]. In the paper [6], S. Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1).

Let both E_1 and E_2 real vector spaces. K. Jun and H. Kim [12] proved that a mapping $f : E_1 \to E_2$ satisfies the functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.2}$$

if and only if there exists a mapping $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x.x.x)$ for all $x \in E_1$, where B, defined by

$$\begin{aligned} B(x,y,z) \;=\; & \frac{1}{24}\big[f(x+y+z)+f(x-y-z) \\ & -f(x+y-z)-f(x-y+z)\big] \end{aligned}$$

for all $x, y, z \in E_1$, is symmetric for each fixed one variable and additive for each fixed two variables. It is easy to see that the functional equation (1.2) is equivalent to a cubic functional equation

$$C(2x+y)+C(x-y)+3C(y)=3C(x+y)+6C(x)$$

and every solution of the cubic functional equation is said to be a cubic mapping [19]. A. Najati [17] investigated the following generalized cubic functional equation:

$$\begin{aligned} & f(kx+y)+f(kx-y) \\ & = kf(x+y)+kf(x-y)+2(k^3-k)f(x) \end{aligned} \tag{1.3}$$

for a positive integers $k \geq 2$.

Now, we introduce the following more generalized functional equation

$$\begin{aligned} & f(rx+sy)+f(rx-sy) \\ & = rs^2f(x+y)+rs^2f(x-y)+2r(r^2-s^2)f(x) \end{aligned} \tag{1.4}$$

where $r \neq -1, 0, 1$ and $s \in \mathbb{R}$. It is easy to see that the function $f(x) = cx^3$ is a solution of the above functional equation. And if one take $r = 2$ and $s = 1$ in (1.4), then the functional equation is (1.2). Also if one take $r \geq 2$ an integer and $s = 1$ in (1.4), then the functional equation is (1.3).

In this paper, we establish the stability problem for the functional equation (1.4) for real number $r \neq -1, 0, 1$ and s in quasi normed spaces and non-Archimedean spaces.

2. The Hyers–Ulam Stability in quasi-Banach spaces

In this section, we investigate the generalized Hyers–Ulam stability problem for the functional equation (1.4) in quasi-Banach space. First, we introduce some basic information concerning quasi-Banach spaces which are referred in [3] and [23]. Let X be a linear space. A quasi-norm is a real-valued function on X satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\lambda x\| = |\lambda| \|x\|$ for any scalar λ and all $x \in X$;
(iii) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible M is called the modulus of concavity of the quasi-norm $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a q-norm$(0 < q \leq 1)$ if $\|x + y\|^q \leq \|x\|^q + \|y\|^q$ for all $x, y \in X$. In this case, a quasi-Banach space is called a q-Banach space. Let X be a quasi-Banach space. Given a q-norm, the formula $d(x, y) := \|x - y\|^q$ gives us a translation invariant metric on X. By Aoki–Rolewicz Theorem [23] (see also [3]), each quasi-norm is equivalent to some q-norm. Since it is much easier to work with q-norms than quasi-norms, here and subsequently, we restrict our attention mainly to q-norms. Moreover, generalized stability theorems of functional equations in quasi-Banach spaces have been investigated by a lot of authors [14, 18, 25].

Now we introduce an abbreviation $D_{r,s}f$ for a given mapping $f : X \to Y$ as follows:

$$\begin{aligned} D_{r,s}f(x,y) \;&:=\; f(rx+sy) + f(rx-sy) \\ &\quad -rs^2 f(x+y) - rs^2 f(x-y) - 2r(r^2-s^2)f(x) \end{aligned}$$

for all $x, y \in X$, where $r \neq -1, 0, 1$ and s are fixed real numbers.

From now on, let X be a normed linear space with quasi-norm $\|\cdot\|$ and Y be a q-Banach space with q-norm $\|\cdot\|$. In this part, by using an direct method, we prove the stability theorem of the equation (1.4).

THEOREM 2.1. *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\sum_{j=0}^{\infty} \frac{1}{|r|^{3jq}} \phi(r^j x, 0)^q < \infty, \quad \lim_{j\to\infty} \frac{\phi(r^j x, r^j y)}{|r|^{3j}} = 0 \tag{2.1}$$

for all $x, y \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\|D_{r,s} f(x,y)\| \le \phi(x,y) \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique mapping $C : X \to Y$ satisfying (1.4) such that

$$\|f(x) - C(x)\| \le \frac{1}{2r^3} \Big[\sum_{j=0}^{\infty} \frac{\phi(r^j x, 0)^q}{|r|^{3jq}} \Big]^{\frac{1}{q}} \tag{2.3}$$

for all $x \in X$.

Proof. Replacing (x, y) by $(x, 0)$ in (2.2), we have

$$\|f(rx) - r^3 f(x)\| \le \frac{1}{2}\phi(x, 0) \tag{2.4}$$

for all $x \in X$. Replacing x by $r^k x$ in (2.4) and then dividing both sides by r^{3k+3}, we get

$$\|\frac{1}{r^{3k}} f(r^k x) - \frac{1}{r^{3k+3}} f(r^{k+1}x)\| \le \frac{1}{2r^3} \frac{\phi(r^k x, 0)}{r^{3k}}$$

for all $x \in X$ and all integers $k \ge 0$. Then for any integers m, k with $m \ge k \ge 0$, we obtain

$$\begin{aligned} &\Big\| \frac{1}{r^{3m+3}} f(r^{m+1}x) - \frac{1}{r^{3k}} f(r^k x) \Big\|^q \\ &= \Big\| \sum_{j=k}^{m} \Big(\frac{1}{r^{3j+3}} f(r^{j+1}x) - \frac{1}{r^{3j}} f(r^j x) \Big) \Big\|^q \\ &\le \sum_{j=k}^{m} \Big\| \frac{1}{r^{3j+3}} f(r^{j+1}x) - \frac{1}{r^{3j}} f(r^j x) \Big\|^q \\ &\le \frac{1}{2^q |r|^{3q}} \sum_{j=k}^{m} \frac{\phi(r^j x, 0)^q}{|r|^{3jq}} \end{aligned} \tag{2.5}$$

for all $x \in X$. Thus the sequence $\Big\{ \dfrac{f(r^k x)}{r^{3k}} \Big\}_{k=1}^{\infty}$ is Cauchy by (2.1). Since Y is complete, this sequence converges for all $x \in X$. So one can define a mapping $C : X \to Y$ by

$$\lim_{k\to\infty} \frac{f(r^k x)}{r^{3k}} = C(x) \quad (x \in X). \tag{2.6}$$

It follows from (2.1) and (2.6) that

$$\begin{aligned} \|D_{r,s}C(x,y)\| &= \lim_{k\to\infty} \frac{1}{r^{3k}} \|D_{r,s}f(r^k x, r^k y)\| \\ &\le \lim_{k\to\infty} \frac{\phi(r^k x, r^k y)}{r^{3k}} = 0 \end{aligned}$$

for all $x, y \in X$. Hence, the mapping C satisfies (1.4). Putting $k := 0$ and letting m go to infinity in (2.5), we see that (2.3) holds. For the uniqueness of C, assume that there exists a mapping $C' : X \to Y$ satisfying (1.4) and (2.3). Then, we find that

$$\begin{aligned} \Big\|C(x) - C'(x)\Big\|^q &= \lim_{k\to\infty} \frac{1}{|r|^{3kq}} \Big\|f(r^k x) - C'(r^k x)\Big\|^q \\ &\le \lim_{k\to\infty} \frac{1}{2^q r^{3q} r^{3kq}} \sum_{j=0}^{\infty} \frac{1}{|r|^{3jq}} \phi(r^{j+k}x, 0)^q \\ &= \frac{1}{2^q |r|^{3q}} \lim_{k\to\infty} \sum_{j=k}^{\infty} \frac{1}{|r|^{3kq}} \phi(r^k x, 0)^q = 0 \end{aligned}$$

for all $x \in X$, which proves the uniqueness. □

THEOREM 2.2. *Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\sum_{j=0}^{\infty} |r|^{3jq} \phi(r^{-j}x, 0)^q < \infty, \quad \lim_{j\to\infty} |r|^{3j} \phi(r^{-j}x, r^{-j}y) = 0$$

for all $x, y, z \in X$. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

$$\|D_{r,s}f(x,y)\| \le \phi(x,y)$$

for all $x, y \in X$. Then there exists a unique mapping $C : X \to Y$ satisfying (1.4) such that

$$\|f(x) - C(x)\| \le \frac{1}{2|r|^3} \Big[\sum_{j=1}^{\infty} |r|^{3jq} \phi(r^{-j}x, 0)^q\Big]^{\frac{1}{q}}$$

for all $x \in X$.

Proof. We observe that one can obtain the following inequality

$$\|r^{3k}f(\frac{x}{r^k}) - r^{3(m+1)}f(\frac{x}{r^{m+1}})\|^q \le \frac{1}{2^q|r|^{3q}}\sum_{j=k}^{m}|r|^{3(j+1)q}\phi(r^{-(j+1)}x,0)^q$$

for all $x \in X$ and all integers k, m with $m \ge k \ge 0$ by use of (2.2). Thus, we see that the proof may be verified by applying similar argument to that of Theorem 2.1. □

In case $r = 2$ and $s = 1$, as a special case of Theorems 2.1 and 2.2 we have the Hyers-Ulam stability results for the cubic functional equation (1.2)(see [12]).

COROLLARY 2.3. *Let $\varepsilon \ge 0$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|D_{r,s}f(x,y)\| \le \varepsilon$$

for all $x, y \in X$. Then there exists a unique mapping $C : X \to Y$ satisfying (1.4) such that

$$\|f(x) - C(x)\| \le \frac{\varepsilon}{2\sqrt[q]{||r|^{3q}-1|}}$$

for all $x \in X$.

COROLLARY 2.4. *Let α, a_1, a_2 be positive real numbers such that either $a_i > 3$ or $a_i < 3$ simultaneously for all $i \in \{1, 2\}$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|D_{r.s}f(x,y)\| \quad \le \quad \alpha\big(\|x\|^{a_1} + \|y\|^{a_2}\big)$$

for all $x, y \in X$. Then there exists a unique mapping $C : X \to Y$ satisfying (1.4) such that

$$\|f(x) - C(x)\| \le \frac{\alpha\|x\|^{a_i}}{2\sqrt[q]{||r|^{3q}-|r|^{q\cdot a_i}|}} \quad (i = 1, 2)$$

for all $x \in X$.

3. The Hyers–Ulam Stability in non-Archimedean spaces

Hensel [10] has introduced a normed space which does not have the non-Archimedean spaces property. During the last three decades, the theory of non-Archimedean spaces has gain the interest of physicists for their research in problems coming from quantum physics, p-adic strings and superstrings [15].

A *valuation* is a function $|\cdot|$ from a field $\mathbb{K}$ to $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|ab| = |a| \cdot |b|$ and the triangle inequality holds, i.e.,

$$|a + b| \leq |a| + |b|, \forall a, b \in \mathbb{K}.$$

A field $\mathbb{K}$ is called a *valued field* if $\mathbb{K}$ equips with a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. Alternatively, if the triangle inequality is replaced by the weakly triangle inequality

$$|a + b| \leq \max\{|a|, |b|\}, \forall a, b \in \mathbb{K},$$

then the valuation $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 3.1. Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a non-Archimedean norm on X if it satisfies the following conditions

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|ax\| = |a|\|x\|$ $(a \in \mathbb{K})$;
(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ $(x, y \in X)$.

In this case $(X, \|\cdot\|)$ is called a non-Archimedean normed space. Because of the fact

$$\|x_k - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq k - 1\} \quad (k > m),$$

a sequence $\{x_m\}$ is Cauchy in the non-Archimedean normed space if and only if $\{x_{m+1} - x_m\}$ converges to zero with respect to the non-Archimedean norm. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Example 3.2. *Let p be a prime number. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on rational $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n_x}^{\infty} a_k p^k| = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field* (see [9, 22]).

Let X be a vector space and Y be a non-Archimedean Banach space. In the following, we now prove the generalized Hyers–Ulam stability of quadratic functional equation (1.4) over the non-Archimedean space. As corollaries, we obtain especially stability result over the p-adic field $\mathbb{Q}_p$. To avoid trivial case, we assume $|r| < 1$.

THEOREM 3.3. *Let $\phi : X^2 \to [0, \infty)$ $\Big(\psi : X^2 \to [0, \infty)\Big)$ be a function such that*

$$\lim_{j\to\infty} \frac{\phi(r^j x, r^j y)}{|r|^{3j}} = 0 \tag{3.1}$$

$$\Big(\lim_{j\to\infty} |r|^{3j}\psi(r^{-j}x, r^{-j}y) = 0, resp\Big)$$

for all $x, y \in X$ and the limit

$$\Phi(x) \equiv \lim_{k\to\infty} \max\Big\{\frac{\phi(r^j x, 0)}{|r|^{3j}} : 0 \le j < k\Big\} \tag{3.2}$$

$$\Big(\Psi(x) \equiv \lim_{k\to\infty} \max\Big\{|r|^{3j}\psi(r^{-j}x, 0) : 1 \le j \le k\Big\}, resp\Big)$$

exists for each $x \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\|D_{r,s}f(x, y)\| \le \phi(x, y) \tag{3.3}$$

$$\Big(\|D_{r,s}f(x, y)\| \le \psi(x, y), resp\Big),$$

for all $x, y \in X$. Then there exists a mapping $C : X \to Y$ satisfying (1.4) such that

$$\|f(x) - C(x)\| \le \frac{1}{|2| \cdot |r|^3}\Phi(x) \tag{3.4}$$

$$\Big(\|f(x) - C(x)\| \le \frac{1}{|2| \cdot |r|^3}\Psi(x), resp\Big),$$

for all $x \in X$. Moreover, if

$$\lim_{m\to\infty}\lim_{k\to\infty} \max\Big\{\frac{\phi(r^j x, 0)}{|r|^{3j}} : m \le j < k + m\Big\} = 0 \tag{3.5}$$

$$\Big(\lim_{m\to\infty}\lim_{k\to\infty} \max\Big\{|r|^{3j}\psi(r^{-j}x, 0) : m < j \le k + m\Big\} = 0, resp\Big),$$

for all $x \in X$, then the mapping C is unique.

Proof. Replacing (x, y) by $(x, 0)$ in (3.3), we have

$$\|f(rx) - r^3 f(x)\| \le \frac{1}{|2|}\phi(x, 0) \tag{3.6}$$

for all $x \in X$. Replacing x by $r^k x$ in (3.6) and then dividing both sides by $|r|^{3k+3}$, we get

$$\|\frac{1}{r^{3k+3}} f(r^{k+1}x) - \frac{1}{r^{3k}} f(r^k x)\| \leq \frac{1}{|2| \cdot |r|^3} \frac{\phi(r^k x, 0)}{|r|^{3k}} \tag{3.7}$$

for all $x \in X$. It follows from (3.7) and (3.1) that the sequence $\{\frac{f(r^k x)}{|r|^{3k}}\}_{k=1}^{\infty}$ is Cauchy in the non-Archimedean Banach space Y. Since Y is complete, we may define a mapping $C : X \to Y$ as $C(x) := \lim_{k\to\infty} \frac{f(r^k x)}{r^{3k}}$ for all $x \in X$. Using induction, one can show that

$$\|\frac{f(r^k x)}{r^{3k}} - f(x)\| \leq \frac{1}{|2| \cdot |r|^3} \max\Big\{\frac{\phi(r^j x, 0)}{|r|^{3j}} : 0 \leq j < k\Big\} \tag{3.8}$$

for all $k \in \mathbb{N}$ and all $x \in X$. By taking k to approach infinity in (3.8) and using (3.2), one obtains (3.4). Replacing x, y and z by $r^{3k}x$,$r^{3k}y$ and $r^{3k}z$, respectively, in (3.3), we get

$$\|\frac{D_{r,s} f(r^k x, r^k y)}{r^{3k}}\| \leq \frac{\phi(r^k x, r^k y)}{|r|^{3k}} \tag{3.9}$$

for all $x, y \in X$. Taking the limit as $k \to \infty$, we conclude that C satisfies (1.4). Moreover, to prove the uniqueness, we assume that there exists a mapping $C' : X \to Y$ satisfying (1.4) and (3.4), (3.5). Then we figure out

$$\begin{aligned}
&\|C(x) - C'(x)\| \\
&= \lim_{m\to\infty} \frac{1}{|r|^{3m}} \|C(r^m x) - C'(r^m x)\| \\
&\leq \lim_{m\to\infty} max\{\frac{\|C(r^m x) - f(r^m x)\|}{|r|^{3m}}, \frac{\|f(r^m x) - C'(r^m x)\|}{|r|^{3m}}\} \\
&\leq \lim_{m\to\infty} \lim_{k\to\infty} \frac{1}{|2| \cdot |r|^3} max\{\frac{\phi(r^j x, 0)}{|r|^{3j}} : m \leq j < m+k\} = 0
\end{aligned}$$

for all $x \in X$. This completes the proof. □

COROLLARY 3.4. *Let X be a non-Archimedean normed space, $t \neq 3$ and θ be positive numbers. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|D_{r,s} f(x, y)\| \leq \theta\big(\|x\|^t + \|y\|^t\big) \quad (x, y \in X).$$

Then there exists a unique mapping $C : X \to Y$ satisfying (1.4) such that

$$\|f(x) - C(x)\| \le \begin{cases} \dfrac{\theta}{|2| \cdot |r|^t}\|x\|^t & if \quad |r| > 1, t > 3 \quad or \quad |r| < 1, t < 3 \\ \dfrac{\theta}{|2| \cdot |r|^3}\|x\|^t & if \quad |r| > 1, t < 3 \quad or \quad |r| < 1, t > 3 \end{cases}$$

for all $x \in X$.

Corollary 3.5. *Let $t \neq 3$ and θ be positive numbers. Suppose that a mapping $f : \mathbb{Q}_p \to \mathbb{Q}_p$ with satisfies the inequality*

$$|D_{p,s}f(x, y)|_p \le \theta\big(|x|_p^t + |y|_p^t\big) \quad (x, y \in \mathbb{Q}_p).$$

Then there exists a unique mapping $C : \mathbb{Q}_p \to \mathbb{Q}_p$ satisfying (1.4) such that

$$|f(x) - C(x)|_p \le \begin{cases} \dfrac{p^t \cdot \theta}{|2|}\|x\|^t & if \quad |r| > 1, t > 3 \quad or \quad |r| < 1, t < 3 \\ \dfrac{p^3 \cdot \theta}{|2|}\|x\|^t & if \quad |r| > 1, t < 3 \quad or \quad |r| < 1, t > 3 \end{cases}$$

for all $x \in \mathbb{Q}_p$.

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