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PARAMETRIC APPROXIMATION OF MONOTONE DECREASING SEQUENCE

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Abstract. The aim of this work is to generalize parametric approximation in order to apply them to an one-sided L_1-approximation.

A natural question now arises : when is the parameter map

$$P : f \to P_{K(f)}(f)$$

continuous on $C_1(X)$ *?*

We find some results with a monotone decreasing sequence about above question.

1. Introduction

Let X be a normed linear space, (V, d) a metric space. For each $p \in V$, let $K(p)$ be a closed convex subset of X. For each $x \in X$, denote by

$$P_{K(p)}(x) : \{y \in K(p) \mid ||x - y|| = d(x, K(p))\},$$

where $d(x, K(p)) = \inf\{||x - y|| \mid y \in K(p)\}$.

Generally, the metric projection depends on x and the approximating set is fixed. In parametric approximation, the metric projection P depends on x, moreover the approximating set $K(p)$ depends on p.

In this paper, we will consider on a space $C_1(X)$, which is a set of all real valued continuous functions on a compact set X with L_1-norm.

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Then the space $C_1(X)$ is not a Banach space but is a dense linear subspace of $L_1(X)$.

Let S be a finite dimensional subspace of $C_1(X)$ and

$$S(f) = \{s \in S \mid s \leq f\}$$

for each $f \in C_1(X)$. Then the set $S(f)$ is closed and convex.

Define a map $r : C(X) \to \mathbb{R}$ is a continuous real-value function on $C_1(X)$ such that for any $f \in C_1(X)$

$$r(f) \geq d(f, S(f)) + ||f||$$

where $d(f, S(f)) = \inf_{s \in S(f)} ||f - s||$.

For each $f \in C_1(X)$, define

$$K(f) = \{s \in S(f) \mid ||s|| \leq r(f)\}$$
$$= S(f) \cap B(0, r(f)).$$

2. Parametric approximation

The Hausdorff metric between any two closed and bounded set A an B is defined as

$$H(A, B) = \max\{h(A, B), h(B, A)\},$$

where $h(A, B) = \sup_{a \in A} d(a, B)$.

It is easy to see that the Hausdorff metric defines a metric on the collection of all nonempty, closed and bounded sets.

LEMMA 2.1. *For each* $f \in C_1(X)$

$$d(f, S(f)) = d(f, K(f))$$

In particular,

$$P_{K(f)}(f) = P_{S(f)}(f).$$

Proof. Let $z \in P_{S(f)}(f)$. Then $z \in S(f)$ and $||z|| \leq r(f)$. So $z \in K(f)$. Hence $d(f, S(f)) = d(f, K(f))$.

If $z \in P_{S(f)}(f)$, then $z \in K(f)$ and $||f - z|| = d(f, K(f))$. Thus $z \in P_{K(f)}(f)$. By the converse proof is similar, $P_{K(f)}(f) = P_{S(f)}(f)$. $\square$

LEMMA 2.2. *For each $f, g \in C_1(X)$*

$$|d(f, K(f)) - d(g, K(g))| = |d(f, S(f)) - d(g, S(g))|$$
$$\leq ||f - g|| + H(K(f), K(g)).$$

Proof. Given $\varepsilon > 0$ there exists $z \in K(f)$ such that

$$d(f, K(f)) + \frac{\varepsilon}{2} \geq ||f - z||.$$

Now there exists $w \in K(g)$ such that $d(z, K(g)) + \frac{\varepsilon}{2} \geq ||z - w||$.

$$\begin{aligned} d(g, K(g)) \leq ||g - w|| &\leq ||g - f|| + ||f - z|| + ||z - w|| \\ &\leq ||f - g|| + d(f, K(f)) + d(z, K(g)) + \varepsilon \\ &\leq ||f - g|| + d(f, K(f)) + h(K(f), K(g)) + \varepsilon \\ &\leq ||f - g|| + d(f, K(f)) + H(K(f), K(g)) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have

$$d(g, K(g)) - d(f, K(f)) \leq ||f - g|| + H(K(f), K(g)).$$

Interchanging the roles of f, g we obtain

$$|d(g, K(g)) - d(f, K(f))| \leq ||f - g|| + H(K(f), K(g)),$$

as desired. $\square$

The following proposition will be very useful in the main result.

PROPOSITION 2.3. *Suppose that $\{f_n\}$ is a monotone decreasing sequence in $C_1(X)$ which converges to $f_0 \in C_1(X)$. For any sequence $\{g_n\}$ with*

$$g_n \in K(f_n) = \{s \in S \mid s \leq f_n,\ ||s|| \leq r(f_n)\}$$

has a subsequence which converges to some element $g_0 \in K(f_0)$.

Proof. Let $\{f_n\}$ be a monotone decreasing sequence in $C_1(X)$ which converges to $f_0 \in C_1(X)$. For each $g_n \in K(f_n)$

$$||g_n|| \leq r(f_n) \quad \text{and} \quad r(f_n) \to r(f_0)$$

there exists a $M > 0$ such that $||g_n|| \leq M$ for all n. Now let $\{h_n\}$ be a subsequence of $\{g_n\}$ which converges to some element $g_0 \in S$.

Suppose that $||g_0|| > r(f_0)$ and let $||g_0|| - r(f_0) = \varepsilon$. Since ε is positive there exists $N_0 \in N$ such that

$$|r(f_n) - r(f_0)| < \varepsilon/2 \quad \text{for any} \quad n > N_0.$$

Since $||h_n|| \leq r(f_n)$

$$||h_n|| - ||g_0|| > \varepsilon/2 \quad \text{for any} \quad n > N_0.$$

It is contradicts.

Since $h_n \leq f_n$ for all n,

$$f_0 - g_0 = \lim(f_n - h_n) \geq 0.$$

At all, $g_0 \in K(f_0)$. □

From Lemma 2.2, we can immediately deduce the following result.

COROLLARY 2.4. *For each $f, g \in C_1(X)$ with $K(f) = K(g)$ then*

$$|d(f, S(f)) - d(g, S(g))| \leq ||f - g||.$$

3. Main result

In this section, using Proposition 2.3 and the two inequalities, we can immediately establish the following continuity property of the parameter map with respect to a monotone decreasing sequence.

THEOREM 3.1. *Suppose that $\{f_n\}$ is a monotone decreasing sequence in $C_1(X)$ which converges to a positive function $f_0 \in C_1(X)$. Then $H(K(f_n), K(f_0)) \to 0$ where the set valued map*

$$\begin{aligned} K : f \to K(f) &= \{g \in S(f) \mid ||g|| \leq r(f)\} \\ &= S(f) \cap B(0, r(f)) \end{aligned}$$

Proof. Let $\{f_n\}$ be a monotone decreasing sequence in $C_1(X)$ which converges to $f_0 \in C_1(X)$. We have to show that

$$\max\{sup_{u\in K(f_n)}d(u,K(f_0)), sup_{v\in K(f_0)}d(v,K(f_n))\} \to 0.$$

If $sup_{u\in K(f_n)}d(u,K(f_0))$ does not converges to 0, then we may assume that given $\varepsilon > 0$ there exists $N_0 \in N, u_n \in K(f_n)$

$$d(u_n, K(f_0)) \geq \varepsilon \quad \text{for each} \quad n \geq N_0.$$

Since $\{u_n\}$ is a sequence in S with $u_n \in K(f_n)$, there exists a convergent subsequence $\{v_n\}$ with $v_n \to v_0 \in K(f_0)$. Thus

$$0 < \varepsilon \leq d(v_n, K(f_0)) \leq ||v_n - v_0|| \to 0$$

for each $n \geq N_0$, which is contradicts. Thus $sup_{u\in K(f_n)}d(u,K(f_0)) \to 0$.

Secondly, suppose that $sup_{v\in K(f_0)}d(v,K(f_n))$ does not converges to 0. For any $\varepsilon > 0$ there exist $N' \in N$ and $v_n \in K(f_0)\cap K(f_n)^c$ such that

$$d(v_n, K(f_n)) \geq \varepsilon \quad \text{for each} \quad n \geq N'.$$

Since $K(f_0)$ is compact, there exists a subsequence $\{u_n\}$ of $\{v_n\}$ with $u_n \to u_0 \in K(f_0)$. If $u_0 = 0$ then

$$0 < \varepsilon \leq d(u_n, K(f_n)) \leq ||u_n|| \to ||u_0|| = 0$$

for each $n \geq N'$. It is a contradiction.

Eventually, $r(f_0) - r(f_n) \leq ||u_0||$, define

$$u_n^* = \frac{(r(f_n) - r(f_0) + ||u_0||)u_n}{||u_n||}.$$

Then $u_n^* \in K(f_n)$ and $u_n^* \to u_n \to u_0$. But

$$\begin{aligned} 0 < \varepsilon \leq d(u_n, K(f_n)) &\leq ||u_n - u_n^*|| \\ &\leq ||u_n - u_0|| + ||u_0 - u_n^*|| \to 0 \end{aligned}$$

for each $n \geq N'$, which is absurd. Hence

$$\max\{sup_{u\in K(f_n)}d(u,K(f_0)), sup_{v\in K(f_0)}d(v,K(f_n))\} \to 0,$$

as desired. $\square$

THEOREM 3.2 ([5]). *Suppose that $K(f)$ is an approximatively compact and convex subset of $C_1(X)$. If the map $f \to K(f)$ is Hausdorff continuous at f_0, then the parameter map*

$$P : f \to P_{K(f)}(f)$$

is upper semicontinuous at f_0.

Combining Theorems 3.5. and 3.6, we recover the following result.

COROLLARY 3.3. *For each $f_0 \in C_1(X)$ the parameter map*

$$P : f \to P_{K(f)}(f)$$

is upper semicontinuous at f_0 with respect to monotone decreasing sequence $\{f_n\}$ which converges to f_0. That is, for any open set O with $P_{K(f_0)}(f_0) \subset O$ there exists $N^* \in N$ such that

$$P_{K(f_n)}(f_n) \subset O \qquad \text{for each } n \geq N^*.$$

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