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ENTROPY MAPS FOR MEASURE EXPANSIVE HOMEOMORPHISMS

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Abstract. It is well known that the entropy map is upper semi-continuous for expansive homeomorphisms on a compact metric space. Recently, Morales [3] introduced the notion of measure expansiveness which is general than that of expansiveness. In this paper, we prove that the entropy map is upper semi-continuous for measure expansive homeomorphisms.

1. Introduction

Let X be a compact metric space with a metric d, and let f be a homeomorphism from X to X.

Definition 1.1. A homeomorphism f is called expansive if there is $\delta > 0$ such that for any distinct point $x, y \in X$ there exists $n \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > \delta$$

Equivalently, a homeomorphism f is *expansive* if there is $\delta > 0$ such that if $d(f^n(x), f^n(y)) \leq \delta$, $n \in \mathbb{Z}$, then $x = y$. Given $x \in X$ and $\delta > 0$, we define $\Gamma_\delta^f(x)$ by

$$\Gamma_\delta^f(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta, \quad for \ \ all \ i \in Z\},$$

and it is called the dynamical δ-ball of f centered at $x \in X$. By definition, it is clear that f is expansive if and only if there exists $\delta > 0$ satisfying $\Gamma_\delta^f(x) = \{x\}$ for all $x \in X$. A Borel measure μ of X said to be non-atomic if $\mu(\Gamma_\delta^f(x)) = 0$, for all $x \in X$.

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Denote by $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with *weak*-topology*, and let $\mathcal{M}^*(X) = \{\mu \in \mathcal{M}(X) : \ \mu \textit{ is nonatomic}\}$.

A homeomorphism $f : X \to X$ of a metric space X is called μ-*expansive* (or μ is expansive for f) if there is $\delta > 0$ such that $\mu(\Gamma_\delta^f(x)) = 0$ for all $x \in X$.

A homeomorphism f is said to be *measure expansive* if f is μ-expansive for all $\mu \in \mathcal{M}^*(X)$.

Let $(X, \mathcal{B}, \mu)$ be a measure space and I a countable family of indices.

Definition 1.2. We say that $\xi = \{C_i : i \in I\} \subset \mathcal{B}$ is a *measure partition* if

(1) $\mu(\bigcup_{i\in I} C_i) = \mu(X)$ and $\mu(C_i) > 0$ for every $i \in I$.
(2) $\mu(C_i \cap C_j) = 0$ for every $i, j \in I$ with $i \notin j$.

We define also $\xi \vee \eta$ by:

$$\xi \vee \eta = \{C \cap D : C \in \xi, D \in \eta, \mu(C \cap D) > 0\}.$$

A strong generator of f is a countable partition ξ which the smallest σ-algebra of $\mathcal{B}$ containing $\bigvee_{k\in\mathbb{N}} f^{-k}(\xi)$ equals $\mathcal{B}$ (mod 0).

Definition 1.3. The *entropy* of a measure partition ξ is given by

$$H_\mu(\xi, X) = -\sum_{C\in\xi} \mu(C) \log \mu(C).$$

Let $f : X \to X$ be a measurable function, and let $\mu \in \mathcal{M}^*(X)$ be f-invariant and ξ is a measure partition of X.

Definition 1.4. The *entropy* of f with respect to μ and ξ is given by:

$$h_\mu(f, \xi, X) = \inf_n \frac{1}{n} H_\mu(\bigvee_{k=0}^{n-1} f^{-k}\xi) = \lim_{n\to=\infty} \frac{1}{n} H_\mu(\bigvee_{k=0}^{n-1} f^{-k}\xi),$$

and the entropy f with respect to μ by:

$$h_\mu(f, X) = \sup\{h_\mu(f, \xi, X) : \xi \text{ is a finite measurable partition}\}.$$

Definition 1.5. Let $f : X \to X$ be a continuous map of a compact metric space X. For $E, F \subset X$ we say that $E(n, \delta)$-*spans* F with respect to f, if for each $y \in F$, there is an $x \in E$ so that $d(f^k(x), f^k(y)) \le \delta$ for all $0 \le k < n$. We let $r_n(F, \delta) = r_n(F, \delta, f)$ denote the minimum cardinality of a set which (n, δ)-spans F. If K is compact, then the continuity of f guarantees $r_n(K, \delta) < \infty$. For compact K, we define

$$\overline{r}_f(K, \delta) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(K, \delta)$$

and

$$h_{top}(f, K) = \lim_{\delta \to 0} \overline{r}_f(K, \delta).$$

Definition 1.6. A homeomorphism $f : X \to X$ said to be *entropy expansive* if there exists $\delta > 0$ such that

$$\sup_{x \in X} \{h_{top}(f, \Gamma_\delta(x))\} = 0,$$

where $\Gamma_\delta(x) = \bigcap_{n \in \mathbb{Z}} f^{-n} B_\epsilon(f^n(x))$.

We have two main results for entropy expansive maps with X compact. First, the topological entropy satisfies $h(f) = h(f, \epsilon)$. Second, assuming X is finite dimensional, $h_\mu(f) = h_\mu(f, \mathbf{A})$ when μ is an f-invariant Borel probability measure on X and $\mathbf{A}$ is a finite measurable partition of X into sets of diameter at most ϵ. These both results are well-known in case f is expansive(See [1] and [4], respectively).

2. Example

The following example shows that the Smale's horseshoe map is measure expansive but it is not entropy expansive.

Example 2.1. (Smale's horseshoe map) Let f be a diffeomorphism on an open neighborhood of the square $Q = [0, 1]^2$. Consider the horizontal strips

$$H_1 = [0, 1] \times [0, a] \text{ and } H_2 = [0, 1] \times [1 - a, 1]$$

and the vertical strips

$$V_1 = [0, a] \times [0, 1] \text{ and } V_2 = [1 - a, 1] \times [0, 1],$$

for some constant $a \in (0, 1/2)$.We assume that

$$f(H_1) = V_1 \text{ and } f(H_2) = V_2 \tag{2.1}$$

which yields the identity

$$Q \cap H(Q) = V_1 \cup V_2. \tag{2.2}$$

We also assume that the restrictions $f|_{H_1}$ are $f|_{H_2}$ are affine, with

$$f(x, y) = \begin{cases} (ax, by) & \text{if } (x, y) \in H_1, \\ (-ax + 1, -by + b) & \text{if } (x, y) \in H_2, \end{cases}$$

where $b = 1/a$. We shall see that the construction of the Smale horseshoe only depends on the restriction $f|_{(H_1 \cup H_2)}$.
Now we consider the diffeomorphism f^{-1}. By (2.2.1), we have

$$f^{-1}(V_1) = H_1 \text{ and } f^{-1}(V_2) = H_2$$

and thus, it follows from (2.2.2) that

$$f^{-1}(Q) \cap Q = f^{-1}(V_1) \cup f^{-1}(V_2) = H_1 \cup H_2. \tag{2.3}$$

Combining (2.2.2) and (2.2.3), we conclude that

$$\bigcap_{k=-1}^{1} f^n(Q) = (H_1 \cup H_2) \cap (V_1 \cup V_2)$$

is the union of 4-square of size a.

Iterating this procedure, that is, considering successively the images $f^n(Q)$ and the preimage $f^{-n}(Q)$, we find that the intersection

$$\Lambda_n = \bigcap_{k=-n}^{n} f^k(Q)$$

is the union of 4^n squares of size a^n. Since Λ_n is a decreasing sequence of nonempty closed sets, the compact set

$$\Lambda = \bigcap_{n \in \mathbb{N}} \Lambda_n = \bigcap_{k \in \mathbb{Z}} f^k(Q)$$

is nonempty. It is called a *Smale horseshoe* (for f).

Clearly, the set Λ has no interior points since the diameter of the 4^n squares in Λ_n tend to zero when $n \to \infty$. One can also verify that Λ has no isolated points. Hence, it is a Cantor set. Therefore, for Lebesque measure m, $m(\Lambda) = 0$. But it is a topologically conjugate to two-sided shift map $\sigma : \Sigma_k \to \Sigma_k$ and its topological entropy is $\log k > 0$.

3. Entropy maps of measure expansive homeomorphisms

Let (X, d) be a compact metric space, and $f : X \to X$ continuous. Let $\mathcal{M}_f^*(X)$ be the space of all probability measures on $(X, \mathcal{B}(X))$ that are f-invariant. We know that $\mathcal{M}_f^*(X)$ is a non-empty convex set which is compact in the weak*-topology, by the Krylov-Bogolubov theorem.

Definition 3.1. The entropy map is $h : \mathcal{M}_f^*(X) \to [0, \infty]$ is given by $\mu \mapsto h_\mu(f)$ for any $\mu \in \mathcal{M}_f^*(X)$.

The entropy map h is affine, *i.e.* if $\mu, m \in \mathcal{M}_f^*(X)$ and $t \in [0,1]$ then

$$h_{t\mu+(1-t)m}(f) = th_\mu(f) + (1-t)h_m(f).$$

Note that the entropy map is not continuous. We will give a counterexample in the case of the two-sided shift on $\{0,1\}^{\mathbb{Z}}$. Let us consider the measures μ_p, for $p \in \mathbb{N}$, concentrated on the p periodic points, giving to each such a point measure $1/2^p$. We have that $\mu_p \in \mathcal{M}_\sigma^*(X)$ and $h_{\mu_p}(\sigma) = 0$, for every p, because the measure is concentrated on a finite set of points. And let μ be the $(1/2, 1/2)$-bernoulli measure, which we know, has $h_\mu(\sigma) = log2$. Now the collection of functions that depends only on a finite number of coordinates form a dense subset $F(X)$ of $C(X)$ by the Stone-Weierstrass theorem. If $f \in F(X)$ then exists N such that $\int_X f d\mu_p = \int_X f d\mu$ if $p \geq N$. Therefore $\mu_p \to \mu$ and so the entropy map is not continuous.

Sometimes it is not even upper semi-continuous, but for a special class of maps we will prove that the entropy map is upper semi-continuous. For instance, it has been shown that entropy map is upper semi-continuous for expansive homeomorphisms of compact metric spaces(for more details, [5]).

Let (X, β) be a measure space. If $f : X \to X$ is measurable and $k \in \mathbb{N}$, we define for every partition P the pullback partition $\{f^{-k}(\xi) : \xi \in P\}$ which is countable.

Definition 3.2. A *measure-sensitive partition* of a measurable map $f : X \to X$ is a countable partition P satisfying

$$\mu(\{y \in X : f^n(y) \in P(f^n(x)), \forall n \in \mathbb{N}\}) = 0 \quad \text{for all } x \in X$$

where $P(x)$ stands for the element of P containing $x \in X$.

The result below is the central motivation of this chapter. By theorem 4.5 in [3], every strong generator of a measurable map f in a non-atomic probability space is a measure-sensitive partition of f. This motivates the question as to whether every measure-sensitive maps has a strong generator.

To concern the next corollary, we give some definition about aperiodicity.

Definition 3.3. we say that a measurable map f is *aperiodic* whenever for all $n \in \mathbb{N}^+$ if $n \in \mathbb{N}^+$ and $f^n(x) = x$ on a measurable set A, then $\mu(A) = 0$ and f is *eventually aperiodic* whenever for all $(n,k) \in \mathbb{N}^+ \times \mathbb{N}$ if A is a measurable set such that for every $x \in A$, there is $0 \leq i \leq k$ such that $f^{n+i}(x) = f^i(x)$, then $\mu(A) = 0$.

It follows easily from the definition that an eventually periodic maps is aperiodic. The converse is true for invertible maps but not in general.

As the above mentioned motivation, we give a partial positive answer for certain maps defined as follows.

We say that f is countable to one(mod 0) if $f^{-1}(x)$ is countable for $\mu - a.e.\ x \in X$ and f is nonsingular if a measurable set A has measure zero if and only if $f^{-1}(A)$ also does. All measure-preserving maps are nonsingular. A Lebesgue probability space is a complete measure space which is isomorphic to the completion of a standard probability space, where standard probability space is Polish borel measure space.

Corollary 3.4 (Cor 2.5., [2]). Every measurable expansive map in a non-atomic separable probability space is measure-expansive.

Corollary 3.5 (Cor 4.17., [3]). The following properties are equivalent for nonsingular countable-to-one (mod 0) maps f on non-atomic Lebesque probability spaces:

(1) f is measure-sensitive.
(2) f is eventually aperiodic.
(3) f is aperiodic.
(4) f has a strong generator.

By using Corollaries 3.4 and 3.5, we concern about continuity of entropy map with measure expansive homeomorphisms. Walter in [5] proved the following theorem for expansive homeomorphisms. We slightly changed the proof provided by Walter to obtain same result for measure expansive homeomorphisms.

Theorem 3.6. *If f is a measure expansive homeomorphism then the entropy map is upper semi-continuous, i.e., if $\mu \in \mathcal{M}_f^*(X)$ and $\epsilon > 0$ then exists a neighborhood U of μ in $\mathcal{M}_f^*(X)$ such that $\nu \in U$ implies that*

$$h_\nu(f) < h_\mu(f) + \epsilon.$$

Proof. Let δ be an expansive constant for f, $\mu \in \mathcal{M}_f^*(X)$ and $\epsilon > 0$. By Cor 2.3.5., there exists a strong generator ξ such that $h_\mu(f) = h_\mu(f, \xi)$. Let N so that

$$\frac{1}{N} H_\mu\left(\bigvee_{k=0}^{N-1} f^{-k}\xi\right) < h_\mu(f) + \epsilon.$$

Fix $\epsilon_1 > 0$ to be chosen later. As μ is regular we choose compact sets:

$$K_{i_0 \cdots i_{N-1}} \subset \bigcap_{k=0}^{N-1} f^{-k} C_{i_k}$$

with $\mu(\bigcap_{k=0}^{N-1} f^{-k} C_{i_k} \backslash K_{i_0 \cdots i_{N-1}} < \epsilon_1$. Then

$$\bigcup_{k=0}^{N-1} \bigcup_{i_k = j} f^k K_{i_0 \cdots i_{N-1}} \subset C_j.$$

The sets $L_j := \bigcup_{k=0}^{N-1} \bigcup_{i_k=j} f^k K_{i_0 \cdots i_{N-1}}$ are compact and disjoint so there is a partition $\eta = \{D_1, \cdots, D_k\}$ with $diam(D_j) < \delta$ and $L_j \subset int(D_j)$. We have

$$K_{i_0 \cdots i_{N-1}} \subset \int \left(\bigcap_{k=0}^{N-1} f^{-k} D_{i_k} \right).$$

By Urysohn's lemma we can choose $f_{i_0 \cdots i_{N-1}} \in C(X)$ such that

- $0 \leq f_{i_0 \cdots i_{N-1}} \leq 1$;
- equals 1 on $K_{i_0 \cdots i_{N-1}}$;
- vanishes on $X \backslash int(\bigcap_{k=0}^{N-1} f^{-k} D_{i_k})$.

Let now

$$U_{i_0 \cdots i_{N-1}} := \left\{ \nu \in \mathcal{M}_f^*(X) : \left| \int f_{i_0 \cdots i_{N-1}} d\nu - \int f_{i_0 \cdots i_{N-1}} d\mu \right| < \epsilon_1 \right\}.$$

The set $U_{i_0 \cdots i_{N-1}}$ is open is $\mathcal{M}_f^*(X)$ and if $\nu \in U_{i_0 \cdots i_{N-1}}$ then

$$\nu \left(\bigcap_{k=0}^{N-1} f^{-k} D_{i_k} \right) \geq \int f_{i_0 \cdots i_{N-1}} d\nu > \int f_{i_0 \cdots i_{N-1}} d\mu - \epsilon_1 \geq \mu(K_{i_0 \cdots i_{N-1}}) - \epsilon_1$$

and

$$\mu \left(\bigcap_{k=0}^{N-1} f^{-k} C_{i_k} \right) - \nu \left(\bigcap_{k=0}^{N-1} f^{-k} D_{i_k} \right) < 2\epsilon_1.$$

Now if $U := \bigcap_{i_0 \cdots i_{N-1}} U_{i_0 \cdots i_{N-1}}$ and $\nu \in U$ then :

$$\left| \mu \left(\bigcap_{k=0}^{N-1} f^{-k} C_{i_k} \right) - \nu \left(\bigcap_{k=0}^{N-1} f^{-k} D_{i_k} \right) \right| < 2\epsilon_1 k^N$$

because if $\sum_{i=1}^n a_i = 1 = \sum_{i=1}^n b_i$ and also exists $c > 0$ with $a_i - b_i < c$ for every i then $|a_i - b_i| < cn \ \forall i$, as $b_i - a_i = \sum_{j \neq i} (a_j - b_j) < cn$.

So if $\nu \in U$ and ϵ_1, then small enough the continuity of $x \log x$ gives :

$$\frac{1}{N} H_\nu \left(\bigvee_{k=0}^{N-1} \left(f^{-k} \eta \right) \right) < \frac{1}{N} H_\mu \left(\bigvee_{k=0}^{N-1} \left(f^{-k} \xi \right) \right) + \frac{\epsilon}{2}. \qquad (*)$$

From $(*)$, we obtain

$$\begin{aligned} h_\nu(f) = h_\nu(f, \eta) &\leq \frac{1}{N} H_\nu \left(\bigvee_{k=0}^{N-1} (f^{-k} \eta) \right) \\ &< \frac{1}{N} H_\mu \left(\bigvee_{k=0}^{N-1} (f^{-k} \xi) \right) + \frac{\epsilon}{2} < h_\mu(f) + \epsilon. \end{aligned}$$

This completes the proof. □

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