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HOMOTOPIC EXTENSION OF HOMOTOPIC MAPS ON ESH–COMPACTIFICATIONS

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Abstract. In this paper, we consider locally compact Hausdorff spaces having the closed unit interval of the real line as the remainder for an ESH-compactification and obtain that in the class of compact maps the extensions of homotopic maps on the respective ESH-compactifications remain homotopic under certain conditions.

1. Introduction

Singular mappings have been extensively used to study compactifications of locally compact spaces ([1, 2, 4, 5, 7]). In particular, a compactification of X having K as a remainder may be constructed by taking as a basis in $X \cup K$ for the open sets of X together with sets of the form $U \cup (f^{-1}(U) - F)$, where $f : X \to K$ is a singular map, U is an arbitrary open set in K and F is an arbitrary compact subset of X. In [3], A. Caterino etc. gave a generalization of this construction that has much wider applicability. This definition which is motivated from that of singular mapping yields a much richer collection of compactifications.

Let X be a locally compact space which is not compact and let $\mathcal{N}$ be the set of non-relatively compact open subsets of X together with ϕ. Let K be a compact space and $\mathcal{B}$ a basis for the open subset of K. Suppose that $\mathcal{B}$ is closed with respect to finite unions, which implies

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that $K \in \mathcal{B}$. Let $\pi : \mathcal{B} \to \mathcal{N}$ be such that $\pi(U) \neq \phi$ for each $U \neq \phi$. We say that π is an essential semilattice homomorphism and denoted by ESH if it satisfies the following conditions:

ESH1. $X - \pi(K)$ is compact.

ESH2. $\pi(U \cup V) \triangle (\pi(U) \cup \pi(V))$ is relatively compact for all $U, V \in \mathcal{B}$.

ESH3. If $U, V \in \mathcal{B}$ and $\overline{U} \cap \overline{V} = \phi$, then $\pi(U) \cap \pi(V)$ is relatively compact.

A lattice homomorphism satisfies all these properties. The use of the word *essential* is to indicate that the properties, which are shared by a lattice homomorphism, hold except for a negligible set. In the study of compactifications of a space X, the compact subsets of X are negligible. We observe that if π is an ESH and $\phi \in \mathcal{B}$ then $\pi(\phi)$ is relatively compact by ESH3 and so $\pi(\phi) = \phi$. Moreover, it follows from ESH2 that if $U, V \in \mathcal{B}$, $U \subset V$ then $\pi(U) - \pi(V)$ is relatively compact. If π is an ESH then there is a natural way to define a topology on the disjoint union of X and K. Let

$$S = \{W \subset X \cup K \mid W = U \cup (\pi(U) - F), U \in \mathcal{B}, F \subset X, F \text{ compact}\}$$

and let $\mathcal{A} = \mathcal{T} \cup S$, where $\mathcal{T}$ is a topology on X and $\mathcal{A}$ is a basis for a topology on $X \cup K$. The only thing in need of proof is that for $W_1, W_2 \in \mathcal{A}$ and each $y \in W_1 \cap W_2$ with $y \in K$, there exists $W_3 \in \mathcal{A}$ such that $y \in W_3 \subset W_1 \cap W_2$. To this end, let $W_1 = U_1 \cup (\pi(U_1) - F_1)$ and $W_2 = U_2 \cup (\pi(U_2) - F_2)$ and let $U_3 \in \mathcal{B}$ be such that $y \in U_3 \subset U_1 \cap U_2$. Both $\pi(U_3) - \pi(U_1)$ and $\pi(U_3) - \pi(U_2)$ are relatively compact. This implies $\pi(U_3) \subset \pi(U_1) \cup G_1$ and $\pi(U_3) \subset \pi(U_2) \cup G_2$, where G_1, G_2 are compact subsets of X. Let $F = F_1 \cup F_2 \cup G_1 \cup G_2$ and $W_3 = U_3 \cup \pi(U_3) - F$. It is clear in this case that one has $y \in W_3 \subset W_1 \cap W_2$ and $W_3 \in S$. It is clear that for each basis $\mathcal{V}$

of X, $S \cup \mathcal{V}$ is a basis for $X \cup_\pi K$. With this topology, $X \cup_\pi K$ is a compact Hausdorff space containing X as a dense subspace.

Note that every singular compactification is an ESH-compactification. If $\mathcal{B}$ is a basis for the topology of K, then we can put $\pi(U) = f^{-1}(U)$.

Throughout this paper, spaces are locally compact Hausdorff.

Recall that a map $f : X \to Y$ is called *compact* if $f^{-1}(K)$ is compact for each compact set K of Y, and that two maps $f, g : X \to Y$ are called *homotopic* if there is a continuous map $H : X \times I \to Y$ satisfying $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

For terms and notations not explained here, we refer to [1, 2, 4, 5, 7, 9].

2. Extension of continuous maps on ESH-compactifications

This section is devoted to prove a result regarding the extension of continuous maps on ESH-compactification of locally compact Hausdorff spaces.

PROPOSITION 2.1. *Let X, Y be locally compact spaces and $f : X \to Y$ a compact map. Let p_1, p_2 be ESH responsible for the ESH-compactifications $X +_{p_1} I$ and $X +_{p_2} I$, respectively, satisfying $f^{-1}(p_2(U)) = p_1(U)$. Then the map $Cf : X +_{p_1} I \to Y +_{p_2} I$, defined by*

$$Cf(x) = \begin{cases} f(x) & \text{if} \quad x \in X \\ x & \text{if} \quad x \in I, \end{cases}$$

is continuous.

Proof. To show that the map $Cf : X +_{p_1} I \to Y +_{p_2} I$ is continuous, take an open subset U of Y. Then $Cf^{-1}(U) = f^{-1}(U)$ which is open in X and hence in $X +_{p_1} I$. For an open set of the form $U \cup (p_2(U) - F$, where U is open in I and F is a compact subset of Y, we have

$$\begin{aligned} Cf^{-1}(U \cup (p_2(U) - F)) &= Cf^{-1}(U) \cup Cf^{-1}(p_2(U) - F) \\ &= U \cup p_1(U) - f^{-1}(F), \end{aligned}$$

which is an open subset of $X +_{p_1} I$. □

3. Extension of maps and homotopy

In this section, we show that in the class of compact maps, the extension of homotopic maps between two locally compact spaces remains homotopic on their respective ESH-compactification under certain conditions, where the remainder for both the spaces is homeomorphic to I.

LEMMA 3.1. *Let X be a locally compact space and let $p : \mathcal{B}_1 \to \mathcal{N}_1$ be an ESH with Y the corresponding ESH remainder of X. Then the correspondence $\pi : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{N}$, defined by $\pi(U \times V) = p(U) \times V$, is an ESH. Here $\mathcal{B}_2$ denotes a basis for I and $\mathcal{N}$ denotes the set of non-relatively compact subsets of $Y \times I$.*

Proof. It is easy to prove the lemma. □

PROPOSITION 3.2. *Let I be an ESH-remainder for X and $p : \mathcal{B} \to \mathcal{N}$ an ESH. Then the space $(X +_p I) \times I$ is homeomorphic to $(X \times I) +_\pi I \times I$.*

Proof. We denote by $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, the topologies on the set $(X +_p I) \times I = X \times I \cup I \times I$ induced by the ESH π and the product topology of $X +_p I$ and I, respectively.

Let $U_1 \times U_2 \in \mathcal{T}_1$, where U_1 and U_2 are open subsets of X and I, respectively. Then U_1 is an open subset of $X +_p I$. Hence $U_1 \times U_2$ is an open subset of $(X +_p I) \times I$, i.e., $U_1 \times U_2 \in \mathcal{T}_2$.

Next, let $U_1 \times U_2 - \pi(U_1 \times U_2) - F \in \mathcal{T}_1$, where $U_1 \times U_2$ is an open subset of $I \times I$ and F is a compact subset of $X \times I$. Then

$$(U_1 \times U_2) \cup \pi((U_1 \times U_2) - F) = U_1 \times U_2 \cup p(U_1) \times U_2 - F,$$

which is an element of $\mathcal{T}_2$. Hence every basis element of $\mathcal{T}_1$ is an element of $\mathcal{T}_2$, i.e., $\mathcal{T}_1 \subset \mathcal{T}_2$.

Since both the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ make the set $X \times I \cup I \times I$ a compact Hausdorff space, we have $\mathcal{T}_2 \subset \mathcal{T}_1$, or $\mathcal{T}_1 = \mathcal{T}_2$. □

PROPOSITION 3.3. *Let X, Y be locally compact spaces, and let $f, g : X \to Y$ be compact maps. Let $H : X \times I \to Y$ be a compact homotopy satisfying $H^{-1}(p_2(U)) = (p_1 \times I)(U \times I)$. Then Cf is homotopic to Cg.*

Proof. Define a map $CH : (X +_{p_1} I) \times I \to Y +_{p_2} I$ by

$$CH(x,t) = \begin{cases} H(x,t) & \text{if} \quad x \in X \\ x & \text{if} \quad x \in I. \end{cases}$$

Then CH is continuous: Take an open subset U of Y. $CH^{-1}(U) = H^{-1}(U)$, which is open in $X \times I$, and hence open in $(X +_{p_1} I) \times I$.

Next, let $U \cup (p_2(U) - K)$ be an open subset of $Y +_{p_2} I$, where U is an open subset of I and K is a compact subset of Y. Then

$$\begin{aligned} CH^{-1}(U \cup (p_2(U) - K)) &= CH^{-1}(U) \cup CH^{-1}((p_2(U)) - CH^{-1}(K) \\ &= (U \times I) \cup (p_1 \times I)(U \times I) - H^{-1}(K) \\ &= (U \times I) \cup \pi(U \times I) - H^{-1}(K), \end{aligned}$$

which is open in $(X +_{p_1} I) \times I$.

Therefore, Cf is homotopic to Cg. □

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