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EUROPEAN CONTINGENT CLAIMS VALUATION UNDER REGIME SWITCHING USING THE MELLIN TRANSFORM APPROACH

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Abstract. In this paper we investigate the pricing of European contingent claims under regime switching. We use the Mellin transform to derive an analytic form of the valuation of contingent claims.

1. Introduction

Valuation of contingent claims under regime switching has been studied actively since discrete jumps in financial variables can be captured through the regime switching model. Naik [6] provided an analytic solution to the price of contingent claims when asset return volatilities are subject to discrete shifts. In the paper, the conditional density of the occupational time of the volatility process is used to obtain an analytic form of the price of contingent claims. Boyle and Draviam [1] derived the coupled partial differential equations to the prices of the contingent claims under regime switching and suggested a finite difference scheme along with an iterative method for the valuation of contingent claims under regime switching.

For the valuations of financial derivatives, the Mellin transform approaches can be found in [2, 3, 4, 5, 7, 8]. Panini and Srivastav [7] first introduced the methodology for option valuation with Mellin transform techniques. They derived the integral equation forms for valuation of European and American plain vanilla/basket put options using the Mellin transform. Jódar et al. [5] proposed the similar method for the Black-Scholes equation using the Mellin transform. The American call

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option and its free boundary problem (Frontczak and Schöbel [4]), the American perpetual put option model (Panini and Srivastav [8]), the stochastic volatility model for the European plain vanilla option (Frontczak [2]), and the jump diffusion model (Frontczak [3]) are also investigated using Mellin transform techniques.

In this paper we use the Mellin transform to derive an analytic form of the valuation of European contingent claims under regime switching. Our methodology does not use the conditional density of the occupational time of the volatility process nor does it use an iterative method to solve the coupled differential equations. Instead, we solve a system of differential equations which can be solved analytically by applying the inverse of the Mellin transform to the solutions and arriving at the valuation of the contingent claims under regime switching.

The rest of this paper is organized as follows. In Section 2, we apply the Mellin transform to the coupled Black-Scholes equations and obtain a new system of differential equations. Section 3 gives an analytic solution of the price of European contingent claims under regime switching.

2. Differential equations and the Mellin transform

We assume that the price process of the underlying asset $S(t)$ evolves according to the following equation

$$\frac{dS(t)}{S(t)} = rdt + \sigma(t)dB(t),$$

where r is the risk free interest rate and $B(t)$ is the standard Brownian motion under the risk neutral probability measure $\mathbb{Q}$. $\sigma(t)$ takes two values σ_1 and σ_2, corresponding to a hidden Markov process $y(t)$ which takes two values $0, 1$ and generated by

$$\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}.$$

We assume that $y(t)$ is independent of the standard Brownian motion $B(t)$.

If we denote $V_k = V_k(S,t),\ S = S(t),\ k = 0, 1$, the price of a contingent claim at time t and in the state $y(t) = k$ that pays $f(S(T))$ at the maturity T is given by

$$V_k(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[f(S(T)|y(t) = k\right].$$

As in Boyle and Draviam [1], we obtain the following coupled partial differential equations (PDEs)

$$(2.1)\quad \begin{cases} \dfrac{\partial V_k}{\partial t} + \dfrac{1}{2}\sigma_k^2 S^2 \dfrac{\partial^2 V_k}{\partial S^2} + rS\dfrac{\partial V_k}{\partial S} - rV_k + \lambda_k(V_{1-k} - V_k) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < S < \infty,\ 0 \le t < T \\ V_k(S,T) = V_{1-k}(S,T) = f(S), \end{cases}$$

for $k = 0, 1$.

Now, we use the definition and some properties of the Mellin transform in Jódar et al. [5] to obtain the analytic solutions to the coupled PDEs (2.1). We assume that $V_k(S,t), \dfrac{\partial V_k(S,t)}{\partial t}, \dfrac{\partial V_k(S,t)}{\partial S}$ and $\dfrac{\partial^2 V_k(S,t)}{\partial S^2}$, $k = 0, 1$, are Mellin transformable. If we denote the Mellin transform of $V_k(S,t)$ by $v_k(t)(z)$ then

$$v_k(t)(z) = \int_0^\infty S^{z-1} V_k(S,t)dS,$$

where z is a complex variable. Also we have the following properties (see [5]):

$$\int_0^\infty S^{z-1}\frac{\partial V_k(S,t)}{\partial t}dS = \frac{\partial}{\partial t}v_k(t)(z),$$
$$\int_0^\infty S^{z}\frac{\partial V_k(S,t)}{\partial S}dS = -zv_k(t)(z),$$
$$\int_0^\infty S^{z+1}\frac{\partial^2 V_k(S,t)}{\partial S^2}dS = (z^2+z)v_k(t)(z).$$

We also assume that $f(S)$ is Mellin transformable and continuous. Let $f^*(z)$ be the Mellin transform of $f(S)$, that is,

$$f^*(z) = \int_0^\infty S^{z-1} f(S)dS.$$

The inverse Mellin transform of $f^*(z)$ is given by

$$f(S) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} S^{-z} f^*(z)dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{-(\alpha+i\tau)} f^*(\alpha+i\tau)d\tau.$$

Applying the Mellin transform to the equations (2.1) gives us the following system of differential equations

$$(2.2)\qquad \begin{cases} \dfrac{\partial}{\partial t}v_k(t)(z) + p_k(z)v_k(t)(z) + \lambda_k v_{1-k}(t)(z) = 0, \\ v_k(T)(z) = v_{1-k}(T)(z) = f^*(z), \end{cases}$$

for $k=0,1$, where

$$p_k(z)=\frac{1}{2}\sigma_k^2 z^2+\Big(\frac{1}{2}\sigma_k^2-r\Big)z-(r+\lambda_k).$$

3. Solutions to the equations

In this section we derive an analytic form of the price of contingent claims under regime switching.

THEOREM 3.1. *For* $0\le t\le T$

$$\begin{cases} v_k(t)(z)=A_k(z)f^*(z)e^{-m_1(z)(T-t)}+B_k(z)f^*(z)e^{-m_2(z)(T-t)},\\ v_{1-k}(t)(z)=A_{1-k}(z)f^*(z)e^{-m_1(z)(T-t)}+B_{1-k}(z)f^*(z)e^{-m_2(z)(T-t)},\end{cases}$$

where

$$(3.1)\quad \begin{cases} A_k(z)=\dfrac{m_2(z)+p_k(z)+\lambda_k}{m_2(z)-m_1(z)},\\ B_k(z)=\dfrac{m_1(z)+p_k(z)+\lambda_k}{m_1(z)-m_2(z)},\\ A_{1-k}(z)=-\dfrac{m_1(z)+p_k(z)}{\lambda_k}\cdot\dfrac{m_2(z)+p_k(z)+\lambda_k}{m_2(z)-m_1(z)},\\ B_{1-k}(z)=-\dfrac{m_2(z)+p_k(z)}{\lambda_k}\cdot\dfrac{m_1(z)+p_k(z)+\lambda_k}{m_1(z)-m_2(z)},\\ m_1(z)=\dfrac{-\Big(p_0(z)+p_1(z)\Big)-\sqrt{\Big(p_0(z)-p_1(z)\Big)^2+4\lambda_0\lambda_1}}{2},\\ m_2(z)=\dfrac{-\Big(p_0(z)+p_1(z)\Big)+\sqrt{\Big(p_0(z)-p_1(z)\Big)^2+4\lambda_0\lambda_1}}{2},\end{cases}$$

for $k=0,1$. *The price of contingent claims at time* t *in the state* k *is given by*

$$\begin{aligned}V_k(S,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}&S^{-(\alpha+i\tau)}\Big\{A_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_1(\alpha+i\tau)(T-t)}\\ &+B_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_2(\alpha+i\tau)(T-t)}\Big\}d\tau \qquad (3.2)\end{aligned}$$

for $k=0,1$.

Proof. From (2.2), we have

$$(3.3)\qquad v_{1-k}(t)(z)=-\frac{1}{\lambda_k}\left(\frac{\partial}{\partial t}v_k(t)(z)+p_k(z)v_k(t)(z)\right)$$

and

$$\frac{\partial}{\partial t}v_{1-k}(t)(z) + p_{1-k}(z)v_{1-k}(t)(z) + \lambda_{1-k}v_k(t)(z) = 0. \tag{3.4}$$

Inserting (3.3) into (3.4) yields

$$\frac{\partial^2}{\partial t^2}v_k(t)(z)+\Big(p_0(z)+p_1(z)\Big)\frac{\partial}{\partial t}v_k(t)(z)+\Big(p_0(z)p_1(z)-\lambda_0\lambda_1\Big)v_k(t)(z) = 0. \tag{3.5}$$

The auxiliary equation of the coupled differential equations (3.5) is given by

$$m^2 + \Big(p_0(z) + p_1(z)\Big)m + \Big(p_0(z)p_1(z) - \lambda_0\lambda_1\Big) = 0,$$

which has two distinct roots $m_1(z), m_2(z)$ are given in (3.1). Then the general solutions to the differential equations (3.5) are

$$v_k(t)(z) = A_k(z)f^*(z)e^{-m_1(z)(T-t)} + B_k(z)f^*(z)e^{-m_2(z)(T-t)}. \tag{3.6}$$

From (3.3) and (3.6), we have

$$\begin{aligned} v_{1-k}(t)(z) = &-\frac{m_1(z)+p_k(z)}{\lambda_k}A_k(z)f^*(z)e^{-m_1(z)(T-t)} \\ &-\frac{m_2(z)+p_k(z)}{\lambda_k}B_k(z)f^*(z)e^{-m_2(z)(T-t)}. \end{aligned}$$

By the terminal conditions of (2.2), we have

$$\begin{cases} A_k(z) + B_k(z) = 1, \\ -\dfrac{m_1(z)+p_k(z)}{\lambda_k}A_k(z) - \dfrac{m_2(z)+p_k(z)}{\lambda_k}B_k(z) = 1. \end{cases} \tag{3.7}$$

Then $A_k(z), B_k(z), A_{1-k}(z)$ and $B_{1-k}(z)$ in (3.1) can be easily obtained from (3.7). The inverse Mellin transform of $v_k(t)(z)$ yields

$$\begin{aligned} V_k(S,t) &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} S^{-z}v_k(t)(z)dz \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{-(\alpha+i\tau)}\Big\{A_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_1(\alpha+i\tau)(T-t)} \\ &\qquad + B_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_2(\alpha+i\tau)(T-t)}\Big\}d\tau, \end{aligned} \tag{3.8}$$

for $k=0,1$. At $t=T$ we have

$$\begin{aligned}V_k(S,T) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{-(\alpha+i\tau)}\Big\{A_k(\alpha+i\tau)f^*(\alpha+i\tau)+B_k(\alpha+i\tau)f^*(\alpha+i\tau)\Big\}d\tau\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{-(\alpha+i\tau)}v_k(T,\alpha+i\tau)d\tau\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{-(\alpha+i\tau)}f^*(\alpha+i\tau)d\tau\\ &= f(S)\end{aligned}$$

for $k=0,1$. Therefore, (3.8) satisfies the terminal conditions of (2.1).

From (3.8) we have

$$\begin{aligned}\frac{\partial V_k}{\partial t} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{-(\alpha+i\tau)}\Big\{m_1(\alpha+i\tau)A_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_1(\alpha+i\tau)(T-t)}\\ &\quad + m_2(\alpha+i\tau)B_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_2(\alpha+i\tau)(T-t)}\Big\}d\tau,\\ \frac{\partial V_k}{\partial S} &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}(\alpha+i\tau)S^{-(\alpha+i\tau)-1}\Big\{A_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_1(\alpha+i\tau)(T-t)}\\ &\quad + B_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_2(\alpha+i\tau)(T-t)}\Big\}d\tau,\\ \frac{\partial^2 V_k}{\partial S^2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\alpha+i\tau)(\alpha+i\tau+1)S^{-(\alpha+i\tau)-2}\Big\{A_k(\alpha+i\tau)f^*(\alpha+i\tau)\\ &\quad \times e^{-m_1(\alpha+i\tau)(T-t)} + B_k(\alpha+i\tau)f^*(\alpha+i\tau)e^{-m_2(\alpha+i\tau)(T-t)}\Big\}d\tau,\end{aligned}$$

for $k=0,1$.

Note that

$$\begin{aligned}A_k(z)\Big(m_1(z)+p_k(z)\Big)+\lambda_k A_{1-k}(z)&=0,\\ B_k(z)\Big(m_2(z)+p_k(z)\Big)+\lambda_k B_{1-k}(z)&=0,\\ A_{1-k}(z)\Big(m_1(z)+p_{1-k}(z)\Big)+\lambda_{1-k}A_k(z)&=0,\\ B_{1-k}(z)\Big(m_2(z)+p_{1-k}(z)\Big)+\lambda_{1-k}B_k(z)&=0,\end{aligned}$$

for $k=0,1$. Thus, (3.2) satisfies

$$\frac{\partial V_k}{\partial t}+\frac{1}{2}\sigma_k^2 S^2\frac{\partial^2 V_k}{\partial S^2}+rS\frac{\partial V_k}{\partial S}-rV_k+\lambda_k(V_{1-k}-V_k)=0,$$

for $k=0,1$. Therefore, $V_k(S,t)$, for $k=0,1$, defined by (3.2) are the solutions to the equations (2.1). □

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