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GENERALIZED AFFINE DEVELOPMENTS

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Abstract. The (affine) development of a smooth curve in a smooth manifold M with respect to an arbitrarily given affine connection in the bundle of affine frames over M is well known (cf. S.Kobayashi and K.Nomizu, Foundations of Differential Geometry, Vol.1). In this paper, we get the generalized affine development of a smooth curve x_t ($t \in [0, 1]$) in M into the affine tangent space at x_0 ($\in M$) with respect to a given generalized affine connection in the bundle of affine frames over M.

1. Introduction

The (affine) development of a smooth curve in a smooth manifold M with respect to an arbitrarily given affine connection in the bundle of affine frames over M is well known (cf. [1, 2, 3]).

The purpose of this paper is to get the generalized affine development of a smooth curve x_t ($0 \leq t \leq 1$) in a C^∞ manifold M into the affine tangent space $A_{x_0}(M)$ with respect to a given generalized affine connection in the bundle of affine frames over M.

The main result in this paper is as follows.

Let $L(M)$ and $A(M)$ be the linear frame and the affine frame bundles over an n-dimensional C^∞ manifold M respectively. Let $\tilde{\gamma} : L(M) \hookrightarrow A(M)$ be the homomorphism of $L(M)$ into $A(M)$ with the group homomorphism $\gamma : GL(n; R) \hookrightarrow A(n; R)$, $\bar{\omega}$ an arbitrarily given generalized affine connection in $A(M)$, and $\tilde{\gamma}^\star \bar{\omega} =: \omega + \varphi$ (ω is the connection (form) in $L(M)$ corresponding to $\bar{\omega}$, and φ is the R^n-valued 1-form on $L(M)$ which is corresponding to $\bar{\omega}$). Let $\tau = x_t$ ($0 \leq t \leq 1$) be a C^∞ curve in M, and $\bar{\tau}_0^t$ the generalized affine parallel displacement of the affine

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tangent space $A_{x_t}(M)$ into $A_{x_0}(M)$ with respect to $\bar{\omega}$ in $A(M)$. Then, the generalized affine development $\bar{C}_t = \bar{\tau}_0^t(x_t)$ $(0 \leq t \leq 1)$ of the curve $\tau = x_t$ $(0 \leq t \leq 1)$ in M into $A_{x_0}(M)$ is given as follows:

$$\bar{C}_t = \bar{\tau}_0^t(x_t) = \bar{\tau}_0^t(\dot{x}_t) - \tau_0^t(\dot{x}_t) \quad (0 \leq t \leq 1),$$

where $\dot{x}_t := dx_t/dt$ and τ_0^t is the linear parallel displacement along τ from x_t to x_0 with respect to ω in $L(M)$ which is corresponding to $\bar{\omega}$ in $A(M)$.

2. Preliminaries

In general, when we regard R^n as an affine space, we denote it by A^n. The group $A(n;R)(= GL(n;R) \times R^n)$ of all affine transformations of A^n is represented by the group of all matrices of the form

$$\tilde{a} = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

where $a = (a^i_j)_{i,j} \in GL(n;R)$ and $\xi = (\xi^i)$, $\xi \in R^n$, is a column vector. The element $\widetilde{a}$ maps a point η of A^n into $a\eta + \xi$. We have the following exact sequence:

$$0 \hookrightarrow R^n \overset{\alpha}{\hookrightarrow} A(n;R) \xrightarrow{\beta} GL(n;R) \longrightarrow 1. \tag{2.2}$$

The tangent space $T_x(M)$ of an n-dimensional smooth manifold M at x $(\in M)$, regarded as an affine space, is denoted by $A_x(M)$ and is called the *affine tangent space*. An *affine frame* of the manifold M at x $(\in M)$ consists of a point $p \in A_x(M)$ and a linear frame $(X_1, \ldots, X_n)$ at x; it is denoted by $(X_1, \ldots, X_n; p)$. We denote by $A(M)$ the set of all affine frames of M and define the projection $\tilde{\pi} : A(M) \to M$ by setting $\tilde{\pi}(\tilde{u}) = x$ for every affine frame $\tilde{u}$ at x. Then, $A(M)(M, A(n;R), \tilde{\pi})$ is a principal fiber bundle over M with group $A(n;R)$. We call $A(M)(M, A(n;R), \tilde{\pi})$ the *bundle of affine frames* over M.

Let $L(M)$ be the bundle of linear frames over M. Corresponding to the group homomorphisms $\beta : A(n;R) \to GL(n;R)$ and $\gamma : GL(n;R) \hookrightarrow A(n;R)$, we have principal fiber bundle homomorphisms $\tilde{\beta} : A(M) \to L(M)$ and $\tilde{\gamma} : L(M) \hookrightarrow A(M)$. Namely, $\tilde{\beta} : A(M) \to L(M)$ maps $(X_1, \ldots, X_n; p)$ into $(X_1, \ldots, X_n)$, and $\tilde{\gamma} : L(M) \hookrightarrow A(M)$ maps $(X_1, \ldots, X_n)$ into $(X_1, \ldots, X_n; 0_x)$, where $0_x \in A_x(M)$ is the point corresponding to the origin of $T_x(M)$. In particular, $L(M)$ can be considered as a subbundle of $A(M)$.

A *generalized affine connection* of M is a connection in the principal fiber bundle $A(M)$ of affine frames over M. We denote by R^n the Lie algebra of the vector group R^n. Corresponding to the exact sequence (2.2) of groups, we have the following exact sequence of the Lie algebras:

$$0 \hookrightarrow R^n \hookrightarrow \mathfrak{a}(n;R) \longrightarrow \mathfrak{gl}(n;R) \longrightarrow 0. \tag{2.3}$$

Therefore,

$$\mathfrak{a}(n;R) = \mathfrak{gl}(n;R) + R^n \quad \text{(semidirect sum)}. \tag{2.4}$$

Let $\bar{\omega}$ be the connection form of a generalized affine connection of M. Then $\tilde{\gamma}^\star\bar{\omega}$ is an $\mathfrak{a}(n;R)$-valued 1-form on $L(M)$, where $\tilde{\gamma}^\star\bar{\omega}$ is the pull back of $\bar{\omega}$ by $\tilde{\gamma}$. Let

$$\tilde{\gamma}^\star\bar{\omega} = \omega + \varphi \tag{2.5}$$

be the decomposition corresponding to $\mathfrak{a}(n;R) = \mathfrak{gl}(n;R) + R^n$, so that ω is a $\mathfrak{gl}(n;R)$-valued 1-form on $L(M)$ and φ is an R^n-valued 1-form on $L(M)$. Here φ is a tensorial 1-form on $L(M)$ of type $(GL(n;R), R^n)$ ([2, §5 of Chapter II]), and hence corresponds to a tensor field of type (1,1) on M.

A generalized affine connection (form) $\bar{\omega}$ is called an *affine connection* (*form*) if, in (2.5), the R^n-valued 1-form φ is the canonical 1-form θ on $L(M)$, i.e.,

$$\theta(X) = u^{-1}(\pi_\star(X)) \text{ for } X \in T_u(P), \quad (u \in L(M), P = (L(M)). \tag{2.6}$$

From now on, we denote by $\tilde{\omega}$ and $\bar{\omega}$ affine connections (forms) and generalized affine connections (forms) in the principal fiber bundle $A(M)$ of all affine frames over M respectively.

For later use, we introduce the following lemmas.

Lemma 2.1. ([2, Proposition 3.1, p.127]) *Let $\bar{\omega}$ be a generalized affine connection* (*form*) *on $A(M)$ and let*

$$\tilde{\gamma}^\star\bar{\omega} = \omega + \varphi,$$

where ω is $\mathfrak{gl}(n;R)$-valued and φ is R^n-valued. Then

(1) *The correspondence between the set of all generalized affine connection forms on $A(M)$ and the set of all pairs consisting of a connection form on $L(M)$ and a tensorial 1-form on $L(M)$ of type $(GL(n;R), R^n)$ given by $\bar{\omega} \to (\omega, \varphi)$ is $1:1$.*
(2) *The homomorphism $\tilde{\beta} : A(M) \to L(M)$ maps horizontal subspaces in $A(M)$ into horizontal subspaces in $L(M)$.*

The following lemma is an immediate consequence of Lemma 2.1.

Lemma 2.2. ([2, Theorem 3.3, p.129]) *The principal fiber bundle homomorphism* $\tilde{\beta} : A(M) \to L(M)$ *maps every affine connection on* $A(M)$ *into a linear connection on* $L(M)$*. Moreover, the map, which is defined by* (1) *of Lemma*1.1, *between the set of all affine connections in* $A(M)$ *and the set of all linear connections in* $L(M)$ *is a one-to-one correspondence.*

3. Developments

3.1. Affine developments

Let $\tau = x_t$ $(0 \leq t \leq 1)$ be a smooth curve in M. The *affine parallel displacement* along τ is an affine transformation of the affine tangent space $A_{x_0}(M)$ at x_0 onto the affine tangent space $A_{x_1}(M)$ at x_1 defined by the given affine connection in $A(M)$. Let $\tilde{\tau}_s^t$ be the affine parallel displacement along the curve τ from x_t to x_0. A cross section of M into the affine tangent bundle (associated with $A(M)$) is called a *point field*. Let p be a point field defined along τ so that p_{x_t} is an element of $A_{x_t}(M)$ for each t. Then $\tilde{\tau}_0^t(p_{x_t})$ describes a curve in $A_{x_0}(M)$. We identify the curve $\tau = x_t$ with the trivial point field along τ, that is, the point field corresponding to the zero vector field along τ. Then the *affine development* of the curve τ in M into the affine tangent space $A_{x_0}(M)$ is the curve $\tilde{\tau}_0^t(x_t)$ in $A_{x_0}(M)$, where $\tilde{\tau}_0^t$ is the affine parallel displacement $A_{x_t}(M) \to A_{x_0}(M)$ along τ (in the reversed direction) from x_t to x_0. The following lemma is well known.

Lemma 3.1. ([2, Proposition 4.1, p.131]) *Given a curve* $\tau = x_t$ $(0 \leq t \leq 1)$ *in* M*, set* $Y_t = \tau_0^t(\dot{x}_t)$*, where* τ_0^t *is the parallel displacement with respect to the linear connection* $(form)$ ω $(\tilde{\gamma}^\star\tilde{\omega} = \omega + \theta)$ *along* τ *from* x_t *to* x_0 *and* $\dot{x}_t = dx_t/dt$*. Let* $\tilde{C}_t$ $(0 \leq t \leq 1)$ *be the curve in* $A_{x_0}(M)$ *starting from the origin* (*that is* $\tilde{C}_0 = x_0$) *such that* $d\tilde{C}_t/dt = Y_t$ *for every* t*. Then* $\tilde{C}_t$ *is the affine development of* τ *into* $A_{x_0}(M)$*.*

Proof. Let u_0 be any point in $L(M)$ such that $\pi(u_0) = x_0$ and u_t is the horizontal lift of x_t in $L(M)$ with respect to the linear connection ω. Let $\tilde{u}_t$ be the horizontal lift of x_t in $A(M)$ with respect to the affine connection (form) $\tilde{\omega}$ such that $\tilde{u}_0 = u_0$. Since, by virtue of Lemma 2.1, the homomorphism $\tilde{\beta} : A(M) \to L(M) = A(M)/R^n$ maps $\tilde{u}_t$ into u_t, there is a curve $\tilde{a}_t$ in $R^n \subset A(n;R)$ such that $\tilde{u}_t = u_t\tilde{a}_t$ and $\tilde{a}_0$ is the identity. Then, since $\tilde{a}_t \in R^n \subset A(n;R)$, we can put

$$\tilde{a}_t = \begin{pmatrix} I_n & \tilde{\xi}(t) \\ 0 & 1 \end{pmatrix}, \quad \tilde{a}_t^{-1} = \begin{pmatrix} I_n & -\tilde{\xi}(t) \\ 0 & 1 \end{pmatrix} \tag{3.1}$$

for each t.

Here we shall find a necessary and sufficient condition for $\tilde{a}_t$ in order that $\tilde{u}_t$ be horizontal with respect to the affine connection (form) $\tilde{\omega}$. From Leibniz's formula, we get

$$\dot{\tilde{u}}_t = \dot{u}_t \tilde{a}_t + u_t \dot{\tilde{a}}_t. \tag{3.2}$$

We obtain by virtue of (3.2) and Lemma 2.1

$$\begin{aligned} \tilde{\omega}(\dot{\tilde{u}}_t) &= Ad(\tilde{a}_t^{-1})(\tilde{\omega}(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t \\ &= Ad(\tilde{a}_t^{-1})(\omega(\dot{u}_t) + \theta(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t \\ &= Ad(\tilde{a}_t^{-1})(\theta(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t, \end{aligned} \tag{3.3}$$

where $\tilde{\omega}$ is the affine connection (form) in $A(M)$ and ω is the linear connection (form) in $L(M)$ which is corresponding to $\tilde{\omega}$. Thus, by virtue of (3.1) and (3.3), we get the fact that $\tilde{u}_t$ is horizontal with respect to the affine connection (form) $\tilde{\omega}$ if and only if

$$\theta(\dot{u}_t) = -\dot{\tilde{a}}_t \ \tilde{a}_t^{-1} = \tilde{a}_t \ (d\tilde{a}_t{}^{-1}/dt) = d\tilde{a}_t{}^{-1}/dt. \tag{3.4}$$

Now, in order to obtain the affine development, we assume that the curve $\tilde{u}_t = u_t \tilde{a}_t$ in $A(M)$ is horizontal with respect to the affine connection $\tilde{\omega}$. Then from (3.1) and (3.4), we obtain

$$Y_t := \tau_0^t(\dot{x}_t) = (u_0 \circ u_t{}^{-1})(\dot{x}_t) = u_0(\theta(\dot{u}_t)) = -u_0(d\tilde{\xi}(t)/dt). \tag{3.5}$$

Since $\tilde{a}_t^{-1} \in A(n;R)$ and $u_t{}^{-1}(x_t) \in A^n$, we have from (3.1)

$$\begin{aligned} \tilde{C}_t &= \tilde{\tau}_0^t(x_t) = \tilde{u}_0(\tilde{u}_t^{-1}(x_t)) \\ &= u_0(\tilde{a}_t^{-1}(u_t{}^{-1}(x_t))) = u_0(u_t{}^{-1}(x_t) - \tilde{\xi}(t)) = -u_0(\tilde{\xi}(t)). \end{aligned} \tag{3.6}$$

By the help of (3.5) and (3.6), we obtain $d\tilde{C}_t/dt = \tau_0^t(\dot{x}_t) = Y_t$. □

3.2. Generalized affine developments

As in the subsection 3.1, let $\tau = x_t$ $(0 \le t \le 1)$ be a (piecewise differentiable) curve in M. The *generalized affine parallel displacement* along τ is a generalized affine transformation of the affine tangent space $A_{x_0}(M)$ at x_0 onto the affine tangent space $A_{x_1}(M)$ at x_1 defined by a given generalized affine connection in $A(M)$. Let $\bar{\tau}_s^t$ be the generalized affine parallel displacement along the curve τ from x_t to x_s. In particular, $\bar{\tau}_0^t$ is the generalized affine parallel displacement $A_{x_t}(M) \to A_{x_0}(M)$

along τ (in the reversed direction) from x_t to x_0. The *generalized affine development* of the curve τ in M into the affine tangent space $A_{x_0}(M)$ is the curve $\bar{\tau}_0^t(x_t)$ in $A_{x_0}(M)$. Now we obtain the following theorem.

THEOREM 3.2. *Let $\bar{\omega}$ be an arbitrarily given generalized affine connection in $A(M)$, and let $\tau = x_t$ $(0 \le t \le 1)$ be a smooth curve in M. Let $\bar{\tau}_0^t$ be the parallel displacement of $A_{x_t}(M)$ into $A_{x_0}(M)$ along τ with respect to the generalized affine connection (form) $\bar{\omega}$. Then the generalized affine development $\bar{C}_t = \bar{\tau}_0^t(x_t)$ $(0 \le t \le 1)$ of the curve $\tau = x_t$ $(0 \le t \le 1)$ in M into $A_{x_0}(M)$ is given as follows:*

$$\bar{C}_t = \bar{\tau}_0^t(x_t) = \bar{\tau}_0^t(\dot{x}_t) - \tau_0^t(\dot{x}_t) \quad (0 \le t \le 1),$$

where $\dot{x}_t := dx_t/dt$ and τ_0^t is the linear parallel displacement along τ from x_t to x_0 with respect to ω in $L(M)$ which is corresponding to $\bar{\omega}$ in $A(M)$.

Proof. For the generalized affine connection $\bar{\omega}$ in $A(M)$, $\tilde{\gamma}^\star(\bar{\omega}) = \omega + \varphi$, where ω (resp. φ) is the linear connection (resp. R^n-valued 1-form) on $L(M)$ which is corresponding to $\bar{\omega}$. Let u_0 be any point in $L(M)$ such that $\pi(u_0) = x_0$ and u_t is the horizontal lift of x_t in $L(M)$ with respect to the linear connection ω. Let $\bar{u}_t$ be the horizontal lift of x_t in $A(M)$ with respect to the generalized affine connection (form) $\bar{\omega}$ such that $\bar{u}_0 = u_0$. Since, by virtue of Lemma 2.1, the homomorphism $\tilde{\beta} : A(M) \to L(M) = A(M)/R^n$ maps $\bar{u}_t$ into u_t, there is a curve $\bar{a}_t$ in $R^n \subset A(n;R)$ such that $\bar{u}_t = u_t\bar{a}_t$ and $\bar{a}_0$ is the identity. Then, since $\bar{a}_t \in R^n \subset A(n;R)$, we can put

$$\bar{a}_t = \begin{pmatrix} I_n & \bar{\xi}(t) \\ 0 & 1 \end{pmatrix}, \quad \bar{a}_t^{-1} = \begin{pmatrix} I_n & -\bar{\xi}(t) \\ 0 & 1 \end{pmatrix} \tag{3.7}$$

for each t.

Here we shall find a necessary and sufficient condition for $\bar{a}_t$ in order that $\bar{u}_t$ be horizontal with respect to the generalized affine connection (form) $\bar{\omega}$. From Leibniz's formula, we get

$$\dot{\bar{u}}_t = \dot{u}_t\bar{a}_t + u_t\dot{\bar{a}}_t. \tag{3.8}$$

We obtain by virtue of (3.8) and Lemma 2.1

$$\begin{aligned} \bar{\omega}(\dot{\bar{u}}_t) &= Ad(\bar{a}_t^{-1})(\bar{\omega}(\dot{u}_t)) + \bar{a}_t^{-1}\dot{\bar{a}}_t \\ &= Ad(\bar{a}_t^{-1})(\omega(\dot{u}_t) + \varphi(\dot{u}_t)) + \bar{a}_t^{-1}\dot{\bar{a}}_t \\ &= Ad(\bar{a}_t^{-1})(\varphi(\dot{u}_t)) + \bar{a}_t^{-1}\dot{\bar{a}}_t. \end{aligned} \tag{3.9}$$

Thus, by virtue of (3.7) and (3.9), we get the fact that $\bar{u}_t$ is horizontal with respect to the generalized affine connection (form) $\bar{\omega}$ if and only if

$$\varphi(\dot{u}_t) = -\dot{\bar{a}}_t\ \bar{a}_t^{-1} = \bar{a}_t\ (d\bar{a}_t{}^{-1}/dt) = d\bar{a}_t{}^{-1}/dt. \tag{3.10}$$

Now, in order to obtain the generalized affine development, we assume that the curve $\bar{u}_t = u_t \bar{a}_t$ in $A(M)$ is horizontal with respect to the generalized affine connection $\bar{\omega}$. Then we get

$$\bar{\tau}_0^t(\dot{x}_t) = (\bar{u}_0 \circ \bar{u}_t^{-1})(\dot{x}_t) = (u_0 \circ \bar{a}_t^{-1} \circ u_t{}^{-1})(\dot{x}_t). \tag{3.11}$$

Since $u_t{}^{-1}(\dot{x}_t) \in A^n$ and $\bar{a}_t^{-1} \in A(n;R)$, we get from (3.7)

$$\bar{a}_t^{-1}(u_t{}^{-1}(\dot{x}_t)) = u_t{}^{-1}(\dot{x}_t) - \bar{\xi}(t). \tag{3.12}$$

By virtue of (3.11) and (3.12), we obtain

$$\bar{\tau}_0^t(\dot{x}_t) = \tau_0^t(\dot{x}_t) - u_0(\bar{\xi}(t)). \tag{3.13}$$

On the other hand, we have

$$\bar{C}_t = \bar{\tau}_0^t(x_t) = \bar{u}_0(\bar{u}_t^{-1}(x_t)) = u_0(\bar{a}_t^{-1}(u_t{}^{-1}(x_t))) = u_0(\bar{a}_t^{-1}(0))). \tag{3.14}$$

We get from (3.7) and (3.14)

$$\bar{C}_t = -\bar{u}_0(\bar{\xi}(t)) = -u_0(\bar{\xi}(t)). \tag{3.15}$$

Therefore, by virtue of (3.13) and (3.15), the generalized affine development $\bar{C}_t$ of a curve $\tau = x_t$ $(0 \le t \le 1)$ in M into $A_{x_0}(M)$ is given as follows:

$$\bar{C}_t = \bar{\tau}_0^t(\dot{x}_t) - \tau_0^t(\dot{x}_t), \tag{3.16}$$

where $\bar{\tau}_0^t$ and τ_0^t are parallel displacements along τ from x_t to x_0 with respect to the generalized affine and the linear connections respectively. □

From the proof of Theorem 3.2, we get the following corollary.

Corollary 3.3. *Let $\bar{\omega}$ be an arbitrarily given generalized affine connection in $A(M)$ such that $\tilde{\gamma}^\star\bar{\omega} = \omega + \varphi$. Let $\tau = x_t$ $(0 \le t \le 1)$ be a smooth curve in M, and u_t a horizontal lift of $\tau = x_t$ in $L(M)$ with respect to ω. Let $\bar{u}_t$ be a smooth curve in $A(M)$ such that $\tilde{\pi}(\bar{u}_t) = x_t$ and $\bar{u}_0 = u_0$. Then, $\bar{u}_t$ is the horizontal lift of $\tau = x_t$ in $A(M)$ with respect to $\bar{\omega}$ if and only if, for each t,*

$$\bar{u}_t = u_t\bar{a}_t\ (\bar{a}_t \in R^n \subset A(n;R)) \quad \text{and} \quad \varphi(\dot{u}_t) = d\bar{a}_t^{-1}/dt.$$

Proof. This is clear from (3.10) and Lemma 2.1. □

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.4. *Let $\bar{\omega}$ be an arbitrarily given generalized affine connection in $A(M)$ such that $\tilde{\gamma}^{\star}\bar{\omega} = \omega + \varphi$. Let $\bar{C}_t$ be the generalized affine development of a curve $\tau = x_t$ $(0 \leq t \leq 1)$ in M into $A_{x_0}(M)$. Then*

(i) *if $\dot{x}_t$ is parallel along $\tau = x_t$ with respect to the generalized affine connection $\bar{\omega}$ in $A(M)$, then $\bar{C}_t = \dot{x}_t|_{t=0} - \tau_0^t(\dot{x}_t)$,*

(ii) *if $\dot{x}_t$ is parallel along $\tau = x_t$ with respect to the linear connection ω in $L(M)$, then $\bar{C}_t = \bar{\tau}_0^t(\dot{x}_t) - \dot{x}_t|_{t=0}$.*

References

[1] C. Ehresmann, *Les connexions infinitesimales dans un espace fibre differentiable, Colloque de topologie,* Bruxelles, 1950.

[2] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. I,* Wiley-Interscience, New York, 1963.

[3] K. Nomizu, *Kinematics and differential geometry-Rolling a ball with a prescribed locus of contact,* Tohoku Math. J. **30** (1978), 623-637.

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