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THE RIEMANN-STIELTJES DIAMOND-ALPHA INTEGRAL ON TIME SCALES

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Abstract. In this paper, we define and study the Riemann–Stieltjes diamond-alpha integral on time scales. Many properties of this integral will be obtained. The Riemann–Stieltjes diamond-alpha integral contains the Riemann–Stieltjes integral and diamond-alpha integral as special cases.

1. Introduction

The calculus on time scales was introduced for the first time in 1988 by Hilger [1] to unify the theory of difference equations and the theory of differential equations. It has been extensively studied on various aspects by several authors [2-8].

Two versions of the calculus on time scales, the delta and nabla calculus, are now standard in the theory of time scales [3, 4]. In 2006, the diamond-alpha integral on time scales was introduced by Sheng, Fadag, Henderson, and Davis [10], as a linear combination of the delta and nabla integrals. The diamond-alpha integral reduces to the standard delta integral for $\alpha = 1$ and to the standard nabla integral for $\alpha = 0$. We refer the reader to [9, 10, 11] for a complete account of the recent diamond-alpha integral on time scales. In 2009, the Riemann diamond-alpha integral on time scales, as a more basic notion of diamond-alpha integral, was introduced by A.B. Malinowska and D.F.M. Torres [12]. In this paper we define the Riemann–Stieltjes diamond-alpha integral on time

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scales, which give a common generalization of the Riemann diamond-alpha integral and the Riemann–Stieltjes integral [8]. We also prove the corresponding main theorems of the Riemann–Stieltjes diamond-alpha integral.

2. Preliminaries

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma(t)$ by $\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$ where $\inf \emptyset = \sup\{\mathbb{T}\}$, while the backward jump operator $\rho(t)$ is defined by $\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$ where $\sup \emptyset = \inf\{\mathbb{T}\}$.

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. A point $t \in \mathbb{T}$ is dense if it is right and left dense; isolated if it is right and left scattered. The forward graininess function $\mu(t)$ and the backward graininess function $\eta(t)$ are defined by $\mu(t) = \sigma(t) - t$, $\eta(t) = t - \rho(t)$ for all $t \in \mathbb{T}$ respectively. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^k := \mathbb{T} \setminus \sup \mathbb{T}$, otherwise $\mathbb{T}^k := \mathbb{T}$; if $\inf \mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_k := \mathbb{T} \setminus \inf \mathbb{T}$, otherwise $\mathbb{T}_k := \mathbb{T}$. We set $\mathbb{T}^k_k := \mathbb{T}^k \bigcap \mathbb{T}_k$.

A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is called regulated provided its right-sided limits exist at all right-dense point of $[a, b)_{\mathbb{T}}$ and its left-sided limits exist at all left-dense point of $(a, b]_{\mathbb{T}}$

A function $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^k$ if there exists a number $f^{\Delta}(t)$ such that, for each $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of f at t and we say that f is delta differentiable if f is delta differentiable for all $t \in \mathbb{T}^k$.

A function $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t \in \mathbb{T}_k$ if there exists a number $f^{\nabla}(t)$ such that, for each $\varepsilon > 0$, there exists a neighborhood V of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|$$

for all $s \in V$. We call $f^{\nabla}(t)$ the nabla derivative of f at t and we say that f is nabla differentiable if f is nabla differentiable for all $t \in \mathbb{T}_k$.

Let $t, s \in \mathbb{T}$ and define $\mu_{t,s} := \sigma(t) - s$ and $\eta_{t,s} := \rho(t) - s$. A function $f : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}^k_k$ if there exists a number $f^{\diamondsuit_\alpha}(t)$ such that, for each $\varepsilon > 0$, there exists a neighborhood U of t such that, for all $s \in U$,

$$|\alpha(f(\sigma(t))-f(s))\eta_{t,s}+(1-\alpha)(f(\rho(t))-f(s))\mu_{t,s}-f^{\diamondsuit_\alpha}(t)\mu_{t,s}\eta_{t,s}| \leq \epsilon|\mu_{t,s}\eta_{t,s}|.$$

3. The Riemann-Stieltjes diamond-α integral

A partition of $[a, b]_\mathbb{T}$ is any finite ordered subset

$$P = \{t_0, t_1, \dots, t_n\} \subset [a, b]_\mathbb{T}, \quad \textit{where} \quad a = t_0 < t_1 < \dots < t_n = b.$$

Each partition $P = \{t_0, t_1, \dots, t_n\}$ of $[a, b]_\mathbb{T}$ decomposes it into subintervals $[t_{i-1}, t_i)_\mathbb{T}$, $i = 1, 2, \dots, n$, such that for $i \neq j$ one has $[t_{i-1}, t_i)_\mathbb{T} \cap [t_{j-1}, t_j)_\mathbb{T} = \emptyset$.

By $\mathcal{P}([a, b]_\mathbb{T})$ we denote the set of all partitions of $[a, b]_\mathbb{T}$. Let $P_n, P_m \in \mathcal{P}([a, b]_\mathbb{T})$. If $P_m \subset P_n$ we call P_n a refinement of P_m. If P_n, P_m are independently chosen, then the partition $P_n \cup P_m$ is a common refinement of P_n and P_m. Let $g : [a, b]_\mathbb{T} \to \mathbb{R}$ be a real-valued non-decreasing function on $[a, b]_\mathbb{T}$. For the partition P we define the set

$$g(P) = \{g(a) = g(t_0), g(t_1), \dots, g(t_n) = g(b)\} \subset g([a, b]_\mathbb{T}).$$

The image $g([a, b]_\mathbb{T})$ is not necessarily an interval in the classical sense, because our interval $[a, b]_\mathbb{T}$ may contain scattered points. From now on let $g : [a, b]_\mathbb{T} \to \mathbb{R}$ be always a non-decreasing real function on the considered interval $[a, b]_\mathbb{T}$.

Let $f : [a, b]_\mathbb{T} \to \mathbb{R}$ be a real-valued bounded function on $[a, b]_\mathbb{T}$. We denote

$$\overline{M} = \sup\{f(t) : t \in [a, b)_\mathbb{T}\}, \quad \overline{m} = \inf\{f(t) : t \in [a, b)_\mathbb{T}\},$$

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and for $1 \leq i \leq n$,

$$\overline{M_i} = \sup\{f(t) : t \in [t_{i-1}, t_i)_\mathbb{T}\}, \quad \overline{m_i} = \inf\{f(t) : t \in [t_{i-1}, t_i)_\mathbb{T}\},$$

$$\underline{M_i} = \sup\{f(t) : t \in (t_{i-1}, t_i]_\mathbb{T}\}, \quad \underline{m_i} = \inf\{f(t) : t \in (t_{i-1}, t_i]_\mathbb{T}\},$$

Let $\alpha \in [0, 1]$. The upper Darboux-Stieltjes $\diamond_\alpha$-sum of f with respect to the partition P, denoted by $U(f, g, P)$, is defined by

$$U(f, g, P) = \sum_{i=1}^{n} (\alpha\overline{M_i} + (1 - \alpha)\underline{M_i})(g(t_i) - g(t_{i-1})),$$

while the lower Darboux-Stieltjes $\diamond_\alpha$-sum of f with respect to the partition P, denoted by $L(f, g, P)$, is defined by

$$L(f,g,P) = \sum_{i=1}^{n} (\alpha \overline{m_i} + (1-\alpha)\underline{m_i})(g(t_i) - g(t_{i-1})).$$

Note that

$$\begin{aligned} U(f,g,P) &\le \sum_{i=1}^{n} (\alpha \overline{M} + (1-\alpha)\underline{M})(g(t_i) - g(t_{i-1})) \\ &= (\alpha \overline{M} + (1-\alpha)\underline{M})(g(b) - g(a)) \end{aligned}$$

and

$$\begin{aligned} L(f,g,P) &\ge \sum_{i=1}^{n} (\alpha \overline{m} + (1-\alpha)\underline{m})(g(t_i) - g(t_{i-1})) \\ &= (\alpha \overline{m} + (1-\alpha)\underline{m})(g(b) - g(a)). \end{aligned}$$

Thus, we have:

$$\begin{aligned} &(\alpha \overline{m} + (1-\alpha)\underline{m})(g(b) - g(a)) \\ &\le L(f,g,P) \le U(f,g,P) \le (\alpha \overline{M} + (1-\alpha)\underline{M})(g(b) - g(a)). \end{aligned}$$

DEFINITION 3.1. Let $I = [a, b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$. The upper Darboux-Stieltjes $\diamond_\alpha-$integral from a to b with respect to function g is defined by

$$\overline{\int_a^b} f(t) \diamond_\alpha g(t) = \inf_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} U(f,g,P);$$

The lower Darboux-Stieltjes $\diamond_\alpha-$integral from a to b with respect to function g is defined by

$$\underline{\int_a^b} f(t) \diamond_\alpha g(t) = \sup_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} L(f,g,P).$$

If $\overline{\int_a^b} f(t) \diamond_\alpha g(t) = \underline{\int_a^b} f(t) \diamond_\alpha g(t)$, then we say that f is Riemann–Stieltjes $\diamond_\alpha-$integrable with respect to g on $[a, b]_{\mathbb{T}}$, and the common value of the integrals, denoted by $\int_a^b f(t) \diamond_\alpha g(t)$, is called the Riemann–Stieltjes $\diamond_\alpha-$ integral.

DEFINITION 3.2. Let $I = [a, b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$. The upper Darboux-Stieltjes $\Delta-$integral from a to b with respect to function g is defined by

$$\overline{\int_a^b} f(t) \Delta g(t) = \inf_{P \in \mathcal{P}([a,b]_{\mathbb{T}})} U(f,g,P)$$

where $U(f,g,P)$ denote the upper Darboux-Stieltjes sum of f with respect to the partition P and

$$U(f,g,P)=\sum_{i=1}^{n} M_i(g(t_i)-g(t_{i-1})), M_i=\sup\{f(t):t\in[t_{i-1},t_i)_{\mathbb{T}}\}.$$

The lower Darboux-Stieltjes $\Delta-$integral from a to b with respect to function g is defined by

$$\underline{\int_a^b} f(t)\Delta g(t)=\sup_{P\in\mathcal{P}([a,b]_{\mathbb{T}})} L(f,g,P).$$

where $L(f,g,P)$ denote the lower Darboux-Stieltjes sum of f with respect to the partition P and

$$L(f,g,P)=\sum_{i=1}^{n} m_i(g(t_i)-g(t_{i-1})), m_i=\inf\{f(t):t\in[t_{i-1},t_i)_{\mathbb{T}}\}.$$

If $\overline{\int_a^b} f(t)\Delta g(t)=\underline{\int_a^b} f(t)\Delta g(t)$, then we say that f is $\Delta-$integrable with respect to g on $[a,b]_{\mathbb{T}}$, and the common value of the integrals, denoted by $\int_a^b f(t)\Delta g(t)$, is called the Riemann-Stieltjes $\Delta-$ integral. Similarly, we can give the definition of the Riemann-Stieltjes $\nabla-$ integral.

We can easily get the following two theorems.

THEOREM 3.3. *If $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is Riemann–Stieltjes $\Delta-$integrable and Riemann–Stieltjes $\nabla-$integrable with respect to $g:[a,b]_{\mathbb{T}}\to\mathbb{R}$ on the interval $[a,b]_{\mathbb{T}}$, then it is Riemann–Stieltjes $\diamond_\alpha-$integral with respect to g on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b f(t)\diamond_\alpha g(t)=\alpha\int_a^b f(t)\Delta g(t)+(1-\alpha)\int_a^b f(t)\nabla g(t).$$

THEOREM 3.4. *Let $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is Riemann–Stieltjes $\diamond_\alpha-$integrable with respect to $g:[a,b]_{\mathbb{T}}\to\mathbb{R}$ on the interval $[a,b]_{\mathbb{T}}$.*

(i) *If $\alpha=1$, then f is Riemann–Stieltjes $\Delta-$integrable with respect to g on $[a,b]_{\mathbb{T}}$.*
(ii) *If $\alpha=0$, then f is Riemann–Stieltjes $\nabla-$integrable with respect to g on $[a,b]_{\mathbb{T}}$.*
(iii) *If $0<\alpha<1$, then f is Riemann–Stieltjes $\Delta-$integrable and Riemann–Stieltjes $\nabla-$integrable with respect to g on $[a,b]_{\mathbb{T}}$.*
(iv) *If $g\equiv t$, then the Riemann–Stieltjes $\diamond_\alpha-$integral reduces to the standard diamond-alpha integral.*

The following theorems may be showed in the same way as Theorem 5.5 and Theorem 5.6 in [4] or Theorem 3.5 and Theorem 3.6 in [8].

Theorem 3.5. *Let $L(f,g,P) = U(f,g,P)$ for some $P \in \mathcal{P}([a,b]_{\mathbb{T}})$, then the function f is Riemann–Stieltjes $\diamond_\alpha-$integrable on the interval $[a,b]_{\mathbb{T}}$ with respect to g and*

$$\int_a^b f(t) \diamond_\alpha g(t) = L(f,g,P) = U(f,g,P).$$

Theorem 3.6. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a bounded function on the interval $[a,b]_{\mathbb{T}}$. Then the function f is Riemann–Stieltjes $\diamond_\alpha-$integrable on the interval $[a,b]_{\mathbb{T}}$ with respect to g if and only if for every $\epsilon > 0$ there exists a partition $P \in \mathcal{P}([a,b]_{\mathbb{T}})$ such that $U(f,g,P) - L(f,g,P) < \epsilon$.*

The following Lemma can be found in [8].

Lemma 3.7. *Let $I = [a,b]_{\mathbb{T}}$ be a closed (bounded) interval in $\mathbb{T}$ and let g be continuous on $[a,b]_{\mathbb{T}}$. For every $\delta > 0$ there is a partition $P_\delta = \{t_0, t_1, \dots, t_n\} \in \mathcal{P}([a,b]_{\mathbb{T}})$ such that for each i one has:*

$$g(t_i) - g(t_{i-1}) \leq \delta \quad \text{or} \quad g(t_i) - g(t_{i-1}) > \delta \wedge \rho(t_i) = t_{i-1}.$$

Theorem 3.8. *A bounded function f on $[a,b]_{\mathbb{T}}$ is Riemann-Stieltjes $\diamond_\alpha-$integrable if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ implies*

$$U(f,g,P_\delta) - L(f,g,P_\delta) < \epsilon.$$

Proof. If for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ implies

$$U(f,g,P_\delta) - L(f,g,P_\delta) < \epsilon,$$

then we have that f on $[a,b]_{\mathbb{T}}$ is integrable by Theorem 3.6.

Conversely, suppose that f is Riemann–Stieltjes $\diamond_\alpha-$integrable with respect to g on $[a,b]_{\mathbb{T}}$. If $\alpha = 1$ or $\alpha = 0$ then, f is Riemann–Stieltjes $\Delta-$integrable or $\nabla-$integrable with respect to function g on $[a,b]_{\mathbb{T}}$. Therefore condition holds from [8,Theorem 2.6]. Now, let $0 < \alpha < 1$, f is Riemann–Stieltjes $\diamond_\alpha-$ integrable with respect to function g, then f is Riemann–Stieltjes $\Delta-$integrable or $\nabla-$integrable. According to [8,Theorem 2.6], for each $\epsilon > 0$ there exists $\delta' > 0$ and $\delta'' > 0$ such that $P_{\delta'} \in \mathcal{P}([a,b]_{\mathbb{T}})$, $P_{\delta''} \in \mathcal{P}([a,b]_{\mathbb{T}})$ we have

$$U(f,g,P_{\delta'}) < \overline{\int_a^b} f(t) \diamond_\alpha g(t) + \frac{\epsilon}{2}, \quad \underline{\int_a^b} f(t) \diamond_\alpha g(t) - \frac{\epsilon}{2} < L(f,g,P_{\delta''}).$$

If $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ where $\delta = \min\{\delta', \delta''\}$, then we have

$$\underline{\int_a^b} f(t) \diamond_\alpha g(t) - \frac{\epsilon}{2} < L(f,g,P_\delta) \le U(f,g,P_\delta) < \overline{\int_a^b} f(t) \diamond_\alpha g(t) + \frac{\epsilon}{2}.$$

Because $\underline{\int_a^b} f(t) \diamond_\alpha g(t) = \overline{\int_a^b} f(t) \diamond_\alpha g(t)$, then

$$U(f,g,P_\delta) - L(f,g,P_\delta) < \epsilon.$$

□

The proofs of the following three results are very similar to the proofs of Theorems 3.5, 3.6 and 3.7 in [8] respectively and hence the proofs are omitted.

Theorem 3.9. *Let functions $f_1,\ f_2 : \mathbb{T} \to \mathbb{R}$ be Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to $g : \mathbb{T} \to \mathbb{R}$ on the interval $[a,b]_{\mathbb{T}}$, and α, β be arbitrary real numbers. Then, $\alpha f_1 \pm\ \beta f_2$ is Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to $g : \mathbb{T} \to \mathbb{R}$ on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b (\alpha f_1(t) \pm\ \beta f_2(t)) \diamond_\alpha g(t) = \alpha \int_a^b f_1(t) \diamond_\alpha g(t) \pm \beta \int_a^b f_2(t) \diamond_\alpha g(t).$$

Theorem 3.10. *Let $f : \mathbb{T} \to \mathbb{R}$ be Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to $g_1,\ g_2 : \mathbb{T} \to \mathbb{R}$ on the interval $[a,b]_{\mathbb{T}}$, and α, β be arbitrary real numbers. Then, f is Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to $\alpha g_1 + \beta g_2$ on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b f(t) \diamond_\alpha (\alpha g_1(t) + \beta g_2(t)) = \alpha \int_a^b f(t) \diamond_\alpha g_1(t) + \beta \int_a^b f(t) \diamond_\alpha g_2(t).$$

Theorem 3.11. *Let $a,b,c \in \mathbb{T}$ and $a < b < c$. If $f : \mathbb{T} \to \mathbb{R}$ is bounded on $[a,c]_{\mathbb{T}}$ and $g : \mathbb{T} \to \mathbb{R}$ is non-decreasing on $[a,c]_{\mathbb{T}}$, then*

$$\int_a^c f(t) \diamond_\alpha g(t) = \int_a^b f(t) \diamond_\alpha g(t) + \int_b^c f(t) \diamond_\alpha g(t).$$

Theorem 3.12. *Let $I = [a,b]_{\mathbb{T}}$, where $a,b \in \mathbb{T}$. Every constant function $f : \mathbb{T} \to \mathbb{R}, f(t) \equiv c$, is Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to g on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b f(t) \diamond_\alpha g(t) = c(g(b) - g(a)).$$

Proof. Let $P \in \mathcal{P}([a,b]_{\mathbb{T}})$ and $P = \{t_0, \cdots, t_n\}$. Then we have

$$U(f,g,P) = L(f,g,P) = c \sum_{i=1}^{n} (g(t_i) - g(t_{i-1})) = c(g(b) - g(a)).$$

Hence, $\overline{\int_a^b} f(t) \diamond_\alpha g(t) = \underline{\int_a^b} f(t) \diamond_\alpha g(t) = c(g(b) - g(a))$. □

The following theorem may be proved in much the same way as [4, Theorem 5.18, 5.19, 5.20, 5.21.].

THEOREM 3.13. *Let $I = [a, b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$.*

(i) *Every monotone function f is Riemann–Stieltjes $\diamond_\alpha-$integrable with respect to g on $[a, b]_{\mathbb{T}}$.*
(ii) *Every continuous function f is Riemann–Stieltjes $\diamond_\alpha-$integrable with respect to g on $[a, b]_{\mathbb{T}}$.*
(iii) *Every bounded function f with only finitely many discontinuity points is Riemann–Stieltjes $\diamond_\alpha-$integrable with respect to g on $[a, b]_{\mathbb{T}}$.*
(iv) *Every regulated function f is Riemann–Stieltjes $\diamond_\alpha-$integrable with respect to g on $[a, b]_{\mathbb{T}}$.*

THEOREM 3.14. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. Then, f is Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to g on $[t, \sigma(t)]_{\mathbb{T}}$ and*

$$\int_t^{\sigma(t)} f(s) \diamond_\alpha g(s) = (\alpha f(t) + (1-\alpha) f(\sigma(t)))(g(\sigma(t)) - g(t)).$$

Moreover, if $0 < \alpha \leq 1$ and g is $\diamond_\alpha-$differentiable at t, then

$$\int_t^{\sigma(t)} f(s) \diamond_\alpha g(s) = \mu(t) g^\Delta(t)(\alpha f(t) + (1-\alpha) f(\sigma(t))).$$

Proof. If $t = \sigma(t)$, then the equality is obvious. If $t < \sigma(t)$, then $\mathcal{P}([t, \sigma(t)]_{\mathbb{T}})$ contains only one element given by

$$t = s_0 < s_1 = \sigma(t).$$

Since $[s_0, s_1)_{\mathbb{T}} = \{t\}$ and $(s_0, s_1]_{\mathbb{T}} = \{\sigma(t)\}$, we have

$$\begin{aligned} U(f, g, P) &= L(f, g, P) \\ &= \alpha f(t)(g(\sigma(t)) - g(t)) + (1-\alpha) f(\sigma(t))(g(\sigma(t)) - g(t)). \end{aligned}$$

By Theorem 3.5, f is Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to g on $[t, \sigma(t)]_{\mathbb{T}}$ and

$$\int_t^{\sigma(t)} f(s) \diamond_\alpha g(s) = (\alpha f(t) + (1-\alpha) f(\sigma(t)))(g(\sigma(t)) - g(t)).$$

By [9, Corollary 3.5., Theorem 3.9.], if $0 < \alpha \leq 1$ and g is $\diamond_\alpha-$differentiable at t, then g is Δ differentiable at t and $g(\sigma(t)) - g(t) = \mu(t) g^\Delta(t)$.

□

THEOREM 3.15. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. Then, f is Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to g on $[\rho(t), t]_{\mathbb{T}}$ and*

$$\int_{\rho(t)}^{t} f(s) \diamond_\alpha g(s) = (\alpha f(\rho(t)) + (1-\alpha) f(t))(g(t) - g(\rho(t))).$$

Moreover, if $0 \le \alpha < 1$ and g is $\diamond_\alpha-$differentiable at t, then

$$\int_{\rho(t)}^{t} f(s) \diamond_\alpha g(s) = \eta(t) g^{\nabla}(t)(\alpha f(\rho(t)) + (1-\alpha) f(t)).$$

Proof. If $t = \rho(t)$, then the equality is obvious. If $t > \rho(t)$, then $[\rho(t), t]_{\mathbb{T}}$ contains only one element given by

$$\rho(t) = s_0 < s_1 = t.$$

Since $[s_0, s_1)_{\mathbb{T}} = \{\rho(t)\}$ and $(s_0, s_1]_{\mathbb{T}} = \{t\}$, we have

$$\begin{aligned} U(f, g, P) &= L(f, g, P) \\ &= \alpha f(\rho(t))(g(t) - g(\rho(t))) + (1-\alpha) f(t)(g(t) - g(\rho(t))). \end{aligned}$$

By Theorem 3.5, f is Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to g on $[\rho(t), t]_{\mathbb{T}}$ and

$$\int_{\rho(t)}^{t} f(s) \diamond_\alpha g(s) = (\alpha f(\rho(t)) + (1-\alpha) f(t))(g(t) - g(\rho(t))).$$

By [9,Corollary 3.5.,Theorem 3.9.], if $0 \le \alpha < 1$ and g is $\diamond_\alpha-$ differentiable at t, then g is ∇ differentiable at t and $g(t) - g(\rho(t)) = \eta(t) g^{\nabla}(t)$. □

By the definition of the Riemann-Stieltjes $\diamond_\alpha-$integral, we have the following Corollary:

COROLLARY 3.16. *Let $a, b \in \mathbb{T}$ and $a < b$. Then we have the following:*

(i) *If $\mathbb{T} = \mathbb{R}$, then a bounded function f is Riemann-Stieltjes $\diamond_\alpha-$integrable with respect to g on the interval $[a, b]_{\mathbb{T}}$ if and only if f is Riemann-Stieltjes integrable with respect to g on $[a, b]_{\mathbb{T}}$ in the classical sense. Moreover, then*

$$\int_a^b f(t) \diamond_\alpha g(t) = \int_a^b f(t) dg(t).$$

(ii) *If* $\mathbb{T} = \mathbb{Z}$, *then each function* $f : \mathbb{Z} \to \mathbb{R}$ *is Riemann-Stieltjes* $\diamond_\alpha-$*integrable with respect to function* $g : \mathbb{Z} \to \mathbb{R}$ *on the interval* $[a, b]_{\mathbb{T}}$. *Moreover*

$$\int_a^b f(t) \diamond_\alpha g(t) = \sum_{t=a}^{b-1} (\alpha f(t) + (1-\alpha) f(t+1))(g(t+1) - g(t)).$$

(iii) *If* $\mathbb{T} = h\mathbb{Z}$, *then each function* $f : h\mathbb{Z} \to \mathbb{R}$ *is Riemann-Stieltjes* $\diamond_\alpha-$*integrable with respect to function* $g : h\mathbb{Z} \to \mathbb{R}$ *on the interval* $[a, b]_{\mathbb{T}}$. *Moreover*

$$\int_a^b f(t) \diamond_\alpha g(t) = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}-1} [\alpha f(kh-h) + (1-\alpha) f(kh)](g(kh) - g(kh-h)).$$

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