

## IDENTITIES INVOLVING TRIBONACCI NUMBERS

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ABSTRACT. The  $kt + r$  subscripted tribonacci numbers will be expressed by three  $k$  step apart tribonacci numbers for any  $0 < r < k \in \mathbb{Z}$ .

### 1. Introduction

A tribonacci sequence  $\{T_n\}_{n \geq 0}$  is a kind of generalization of fibonacci sequence that begins with three initials  $T_0 = 0$  and  $T_1 = T_2 = 1$  and each next is the sum of the previous three terms. So,  $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ , and the first few terms are  $0, 1, 1, 2, 4, 7, \dots$ . The sequence can be extended to negative  $n$  that  $T_{-1} = 0$ ,  $T_{-2} = 1$ ,  $T_{-3} = -1$ ,  $T_{-4} = 0$ ,  $T_{-5} = 2$ ,  $T_{-6} = -3, \dots$ .

A number of properties of the sequence were studied by many researchers ([2], [3], [4] and [5]). In particular, by means of generating matrices, it was proved in [4] that the  $4t$  subscripted tribonacci numbers satisfy  $T_{4(t+1)} = 11T_{4t} + 5T_{4(t-1)} + T_{4(t-2)}$ , and the sum is  $\sum_{t=0}^n T_{4t} = (T_{4n+4} + 6T_{4n} + T_{4n-4} - T_4)/T_4^2$ .

In this work we shall generalize the identities about  $4t$  subscripted tribonacci numbers  $T_{4t}$  to any  $T_{4t+r}$  ( $1 \leq r \leq 4$ ), so that the properties in [4] are regarded as a case of  $r = 4$ . Moreover the identities will be extended to any  $kt + r$  subscripted tribonacci  $T_{kt+r}$  for any  $1 \leq r \leq k \in \mathbb{Z}$ . One of our main theorem is to express  $T_{kt+r}$  by  $T_{2k+r}$ ,  $T_{k+r}$  and  $T_r$ , which are  $k$  step apart terms.

### 2. Tribonacci table

For  $k \in \mathbb{N}$ , when we say  $k$  columns tribonacci table we mean a rectangle shape having  $k$  columns that consists of the all tribonacci numbers

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from  $T_1$  in order.

$$\begin{array}{cccc} T_1 & T_2 & \cdots & T_k \\ T_{k+1} & T_{k+2} & \cdots & T_{2k} \\ T_{2k+1} & T_{2k+2} & \cdots & T_{3k} \\ \vdots & & \cdots & \vdots \end{array}$$

We shall investigate a third order linear recurrence  $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$  for tribonacci numbers with some  $a_k, b_k \in \mathbb{Z}$ .

LEMMA 2.1.  $T_n = 3T_{n-2} + T_{n-4} + T_{n-6}$ ,  $T_n = 7T_{n-3} - 5T_{n-6} + T_{n-9}$ , and  $T_n = 11T_{n-4} + 5T_{n-8} + T_{n-12}$  for any  $n \in \mathbb{Z}$ .

*Proof.* Observe that  $T_6 = 13 = 3 \cdot 4 + 1 = 3T_4 + T_2 + T_0$ . If we assume  $T_s = 3T_{s-2} + T_{s-4} + T_{s-6}$  for all  $s < n$  then

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3} \\ &= 3(T_{n-3} + T_{n-4} + T_{n-5}) + (T_{n-5} + T_{n-6} + T_{n-7}) \\ &\quad + (T_{n-7} + T_{n-8} + T_{n-9}) \\ &= 3T_{n-2} + T_{n-4} + T_{n-6}. \end{aligned}$$

Similar to this, we notice  $T_9 = 81 = 7 \cdot 13 - 5 \cdot 2 + 0 = 7T_6 - 5T_3 + T_0$ . If we assume  $T_s = 7T_{s-3} - 5T_{s-6} + T_{s-9}$  for all  $s < n$ , then the induction hypothesis proves  $T_n = 7T_{n-3} - 5T_{n-6} + T_{n-9}$ .

Analogously, since  $T_{12} = 504 = 11 \cdot 44 + 5 \cdot 4 + 0 = 11T_8 + 5T_4 + T_0$ , the identity  $T_n = 11T_{n-4} + 5T_{n-8} + T_{n-12}$  can be proved immediately by induction.  $\square$

Note that the identity  $T_{4(t+1)} = 11T_{4t} + 5T_{4(t-1)} + T_{4(t-2)}$  in [4] is a special case of  $T_n = 11T_{n-4} + 5T_{n-8} + T_{n-12}$  in Lemma 2.1 when  $4 \mid n$ . We extend Lemma 2.1 to any integer  $1 \leq k \leq 10$ .

THEOREM 2.2. *Let  $1 \leq k \leq 10$ . Then the third order recurrence  $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$  of  $T_n$  holds with the following  $(a_k, b_k)$ .*

$k$	$(a_k, b_k)$	$T_n = a_k T_{n-k}$	$+ b_k T_{n-2k}$	$+ T_{n-3k}$
1	(1, 1)	$T_n = T_{n-1}$	$+ T_{n-2}$	$+ T_{n-3}$
2	(3, 1)	$T_n = 3 T_{n-2}$	$+ T_{n-4}$	$+ T_{n-6}$
3	(7, -5)	$T_n = 7 T_{n-3}$	$-5 T_{n-6}$	$+ T_{n-9}$
4	(11, 5)	$T_n = 11 T_{n-4}$	$+5 T_{n-8}$	$+ T_{n-12}$
5	(21, 1)	$T_n = 21 T_{n-5}$	$+ T_{n-10}$	$+ T_{n-15}$
6	(39, -11)	$T_n = 39 T_{n-6}$	$-11 T_{n-12}$	$+ T_{n-18}$
7	(71, 15)	$T_n = 71 T_{n-7}$	$+15 T_{n-14}$	$+ T_{n-21}$
8	(131, -3)	$T_n = 131 T_{n-8}$	$-3 T_{n-16}$	$+ T_{n-24}$
9	(241, -23)	$T_n = 241 T_{n-9}$	$-23 T_{n-18}$	$+ T_{n-27}$
10	(443, 41)	$T_n = 443 T_{n-10}$	$+41 T_{n-20}$	$+ T_{n-30}$

*Proof.* Clearly  $T_n = T_{n-1} + T_{n-2} + T_{n-3}$  shows  $(a_1, b_1) = (1, 1)$ . And Lemma 2.1 shows  $(a_k, b_k) = (3, 1)$ ,  $(7, -5)$  and  $(11, 5)$  for  $k = 2, 3, 4$  respectively.

Let  $n = kt + r$  ( $1 \leq r \leq k$ ) and  $5 \leq k \leq 10$ . In order to express  $T_{kt+r} = a_k T_{k(t-1)+r} + b_k T_{k(t-2)+r} + T_{k(t-3)+r}$ , we shall consider  $k$  columns tribonacci tables. Let us begin with  $k = 5$ .

1	1	2	4	7
13	24	44	81	149
274	504	927	1705	3136
5768	10609	19513	35890	...

Then it can be observed that, for instance

$$\begin{cases} T_{16} = 5768 = (21)274 + 13 + 1 = (21)T_{11} + T_6 + T_1 \\ T_{17} = 10609 = (21)504 + 24 + 1 = (21)T_{12} + T_7 + T_2 \\ T_{18} = 19513 = (21)927 + 44 + 2 = (21)T_{13} + T_8 + T_3 \end{cases}$$

Thus, by assuming  $T_s = (21)T_{s-5} + T_{s-10} + T_{s-15}$  for all  $s < n$ , the induction hypothesis gives rise to

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3} \\ &= (21)(T_{n-6} + T_{n-7} + T_{n-8}) + (T_{n-11} + T_{n-12} + T_{n-13}) \\ &\quad + (T_{n-16} + T_{n-17} + T_{n-18}) \\ &= (21)T_{n-5} + T_{n-10} + T_{n-15}, \end{aligned}$$

so  $(a_5, b_5) = (21, 1)$ . Moreover from the 6 columns tribonacci table

1	1	2	4	7	13
24	44	81	149	274	504
927	1705	3136	5768	10609	19513
35890	66012	121415	223317	410744	...

we can observe that, for instance

$$\begin{cases} T_{19} = 35890 = (39)927 - (11)24 + 1 = (39)T_{13} - 11T_7 + T_1 \\ T_{20} = 66012 = (39)1705 - (11)44 + 1 = (39)T_{14} - 11T_8 + T_2 \\ T_{21} = 121415 = (39)3136 - (11)81 + 2 = (39)T_{15} - 11T_9 + T_3 \end{cases}$$

By assuming  $T_s = (39)T_{s-6} - 11T_{s-12} + T_{s-18}$  for all  $s < n$ , we have

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3} \\ &= (39)(T_{n-7} + T_{n-8} + T_{n-9}) - 11(T_{n-13} + T_{n-14} + T_{n-15}) \\ &\quad + (T_{n-19} + T_{n-20} + T_{n-21}) \\ &= (39)T_{n-6} - 11T_{n-12} + T_{n-18}, \end{aligned}$$

so  $(a_6, b_6) = (39, -11)$ . Now from the 7 columns tribonacci table

1	1	2	4	7	13	24
44	81	149	274	504	927	1705
3136	5768	10609	19513	35890	66012	121415
223317	410744	755476	1389537	2555757	4700770	...

we can find the identities that, for instance

$$\begin{cases} T_{22} = 223317 = (71)3136 + (15)44 + 1 = (71)T_{15} + 15T_8 + T_1 \\ T_{23} = 410744 = (71)5768 + (15)81 + 1 = (71)T_{16} + 15T_9 + T_2 \\ T_{24} = 755476 = (71)10609 + (15)149 + 2 = (71)T_{17} + 15T_{10} + T_3, \end{cases}$$

so induction proves that  $T_n = a_7T_{n-7} + b_7T_{n-14} + T_{n-21}$  with  $(a_7, b_7) = (71, 15)$ .

Continuing this way, the 8 columns tribonacci table, as well as 9 columns and 10 columns tribonacci table provides

$$\begin{cases} T_{25} = 1389537 = (131)10609 - (3)81 + 1 = (131)T_{17} - 3T_9 + T_1 \\ T_{26} = 2555757 = (131)19513 - (3)149 + 1 = (131)T_{18} - 3T_{10} + T_2 \\ T_{27} = 4700770 = (131)35890 - (3)274 + 2 = (131)T_{19} - 3T_{11} + T_3, \\ T_{28} = 8646064 = (241)35890 - (23)149 + 1 = (241)T_{19} - (23)T_{10} + T_1 \\ T_{29} = 15902591 = (241)66012 - (23)274 + 1 = (241)T_{20} - (23)T_{11} + T_2 \\ T_{30} = 29249425 = (241)121415 - (23)504 + 2 = (241)T_{21} - (23)T_{12} + T_3, \end{cases}$$

and

$$\begin{cases} T_{31} = 53798080 = (443)121415 + (41)274 + 1 = (443)T_{21} + (41)T_{11} + T_1 \\ T_{32} = 98950096 = (443)223317 + (41)504 + 1 = (443)T_{22} + (41)T_{12} + T_2 \\ T_{33} = 181997601 = (443)410744 + (41)927 + 2 = (443)T_{23} + (41)T_{13} + T_3, \end{cases}$$

respectively. Therefore the observations and mathematical induction prove that the coefficients  $(a_k, b_k)$  for  $8 \leq k \leq 10$  satisfying  $T_n = a_kT_{n-k} + b_kT_{n-2k} + T_{n-3k}$  are equal to  $(131, -3)$ ,  $(241, -23)$ ,  $(443, 41)$ , respectively.  $\square$

We note that the subscript  $n$  of  $T_n$  could be negative, for example, in 6 columns tribonacci table,  $T_{15} = (39)T_9 - 11T_3 + T_{-3} = (39)81 - 11(2) - 1 = 3136$ .

A sequence  $\{t_n\}$  is called a tribonacci type if it satisfies  $t_n + t_{n+1} + t_{n+2} = t_{n+3}$  with any initials  $t_1, t_2$  and  $t_3$ .

**THEOREM 2.3.** *For  $1 \leq k \leq 10$ , let  $(a_k, b_k)$  be the coefficient of the third order recurrence  $T_n = a_kT_{n-k} + b_kT_{n-2k} + T_{n-3k}$ .*

- (1)  $\{a_k\}$  is a tribonacci type sequence  $a_i + a_{i+1} + a_{i+2} = a_{i+3}$  with initials  $a_1 = 1, a_2 = 3$  and  $a_3 = 7$ , while  $\{b_k\}$  satisfies  $b_i - b_{i+1} - b_{i+2} = b_{i+3}$  with  $b_1 = b_2 = 1$  and  $b_3 = -5$ , for  $1 \leq i \leq 7$ .
- (2) Moreover  $a_k = 3T_k - T_{k-6}$  and  $b_k = -a_{-k}$  for  $1 \leq k \leq 10$ .

*Proof.* By Theorem 2.2,  $\{a_k\}_{k=1}^{10} = \{1, 3, 7, 11, 21, 39, 71, 131, 241, 443\}$  and  $\{b_k\}_{k=1}^{10} = \{1, 1, -5, 5, 1, -11, 15, -3, -23, 41\}$ . So it is easy to see that  $a_i + a_{i+1} + a_{i+2} = a_{i+3}$  and  $b_i - b_{i+1} - b_{i+2} = b_{i+3}$  for  $1 \leq i \leq 7$ .

Moreover by means of tribonacci numbers  $T_i$ , we notice

$$a_1 = 1 = 3T_1 - T_{-5}, \quad a_2 = 3 = 3T_2 - T_{-4}, \quad a_3 = 7 = 3T_3 - T_{-3},$$

and  $a_4 = a_3 + a_2 + a_1 = 3T_4 - T_{-2}$ , etc. So  $a_k = 3T_k - T_{k-6}$  for  $1 \leq k \leq 10$ . By considering  $T_k$  with negative  $k$ , the tribonacci type sequence  $\{a_k\}$  can be extended to any  $k \in \mathbb{Z}$ , as follows.

$$\begin{array}{cccccccccccccccc} k | & \cdots & -10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\ \hline a_k | & \cdots & -41 & 23 & 3 & -15 & 11 & -1 & -5 & 5 & -1 & -1 & 3 & 1 & 3 & 7 & \cdots \end{array}$$

Then by comparing  $\{a_k\}_{k=-1}^{-10} = \{-1, -1, 5, -5, -1, 11, -15, 3, 23, -41\}$  with  $\{b_k\}_{k=1}^{10}$ , we find that  $b_k = -a_{-k}$  for  $1 \leq k \leq 10$ . □

### 3. The third order linear recurrence of $T_n$

We shall generalize the findings in Section 2 for  $1 \leq k \leq 10$  to any integer  $k$ .

**THEOREM 3.1.** *Let  $a_k = 3T_k - T_{k-6}$  and  $b_k = -a_{-k}$  for any  $k > 0$ . Then any  $n$ th tribonacci number satisfies  $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$  for every  $k < n$ .*

*Proof.* It is due to Theorem 2.3 if  $1 \leq k \leq 10$ . Since  $a_k = 3T_k - T_{k-6}$  for all  $k$ ,  $\{a_k\}$  is a tribonacci type sequence because

$$\begin{aligned} a_k + a_{k+1} + a_{k+2} &= (3T_k - T_{k-6}) + (3T_{k+1} - T_{k-5}) + (3T_{k+2} - T_{k-4}) \\ &= 3(T_k + T_{k+1} + T_{k+2}) - (T_{k-6} + T_{k-5} + T_{k-4}) \\ &= 3T_{k+3} - T_{k-3} = a_{k+3}. \end{aligned}$$

Similarly, since  $b_k = -a_{-k}$  for all  $k$ ,  $\{b_k\}$  satisfies

$$\begin{aligned} b_k - b_{k+1} - b_{k+2} &= -a_{-k} + a_{-(k+1)} + a_{-(k+2)} \\ &= -(a_{-k-3} + a_{-k-2} + a_{-k-1}) + a_{-k-1} + a_{-k-2} \\ &= -a_{-k-3} = b_{k+3}. \end{aligned}$$

We now suppose that the three order recurrence  $a_i T_{n-i} + b_i T_{n-2i} + T_{n-3i} = T_n$  hold for all  $i \leq k$ . Since

$$\begin{aligned} T_{n-(k-2)} &= T_{n-(k-1)} + T_{n-k} + T_{n-(k+1)}, \\ T_{n-2(k-2)} &= 3T_{n-2(k-1)} + T_{n-2k} + T_{n-2(k+1)}, \\ T_{n-3(k-2)} &= 7T_{n-3(k-1)} - 5T_{n-3k} + T_{n-3(k+1)} \end{aligned}$$

by Lemma 2.1, the mathematical induction with long calculations proves that

$$\begin{aligned} & a_{k+1}T_{n-(k+1)} + b_{k+1}T_{n-2(k+1)} + T_{n-3(k+1)} \\ &= (a_k + a_{k-1} + a_{k-2})(-T_{n-k} - T_{n-(k-1)} + T_{n-(k-2)}) \\ & \quad + (-b_k - b_{k-1} + b_{k-2})(-3T_{n-2(k-1)} - T_{n-2k} + T_{n-2(k-2)}) \\ & \quad + (-7T_{n-3(k-1)} + 5T_{n-3k} + T_{n-3(k-2)}) = T_n. \end{aligned}$$

□

Theorem 3.1 provides a good way to find huge tribonacci numbers. For instance, for 50th tribonacci number  $T_{50}$ , we may choose any  $k$ , say  $k = 11$ , then  $a_{11} = 3T_{11} - T_5 = 3(274) - 7 = 815$  and  $b_{11} = -b_{10} - b_9 + b_8 = -21$ , thus

$$T_{50} = a_{11}T_{50-11} + b_{11}T_{50-22} + T_{50-33} = 5,742,568,741,225,$$

a 13 digit integer. On the other hand, if we take  $k = 12$  then  $a_{12} = 3T_{12} - T_6 = 3 \cdot 504 - 13 = 1499$  and  $b_{12} = -b_{11} - b_{10} + b_9 = -43$ , so  $T_{50}$  can be obtained by  $T_{50} = a_{12}T_{38} + b_{12}T_{26} + T_{14} = 5,742,568,741,225$ .

Besides the expression of  $a_k$  by six step apart tribonacci numbers, more identities for  $a_k$  can be developed in terms of three successive tribonacci numbers.

**THEOREM 3.2.** *Let  $k$  be any integer. Then*

$$a_k = T_k + 2T_{k-1} + 3T_{k-2} = T_{k+1} + T_{k-1} + 2T_{k-2} = 3T_{k+1} - 2T_k - T_{k-1}.$$

$$\text{So } a_k = [3 \quad -2 \quad -1] \begin{bmatrix} T_{k+1} \\ T_k \\ T_{k-1} \end{bmatrix} \text{ and } a_{-k} = [3 \quad -2 \quad -1] \begin{bmatrix} T_{-(k-1)} \\ T_{-k} \\ T_{-(k+1)} \end{bmatrix}.$$

*Proof.* Since

$$\begin{aligned} T_{k-6} &= T_{k-3} - T_{k-4} - (T_{k-2} - T_{k-3} - T_{k-4}) = 2T_{k-3} - T_{k-2} \\ &= 2(T_k - T_{k-1} - T_{k-2}) - T_{k-2} = 2T_k - 2T_{k-1} - 3T_{k-2}, \end{aligned}$$

it follows that

$$a_k = 3T_k - T_{k-6} = 3T_k - 2T_k + 2T_{k-1} + 3T_{k-2} = T_k + 2T_{k-1} + 3T_{k-2}.$$

Hence we have

$$\begin{aligned} a_k &= (T_{k+1} - T_{k-1} - T_{k-2}) + 2T_{k-1} + 3T_{k-2} = T_{k+1} + T_{k-1} + 2T_{k-2} \\ &= T_{k+1} + T_{k-1} + 2(T_{k+1} - T_k - T_{k-1}) = [3 \quad -2 \quad -1] \begin{bmatrix} T_{k+1} \\ T_k \\ T_{k-1} \end{bmatrix}. \end{aligned}$$

Therefore it follows immediately from

$$a_{-k} = T_{-k} + 2T_{-(k+1)} + 3T_{-(k+2)} = T_{-(k-1)} + T_{-(k+1)} + 2T_{-(k+2)}$$

$$= 3T_{-(k-1)} - 2T_{-k} - T_{-(k+1)} = [3 \ -2 \ -1] \begin{bmatrix} T_{-(k-1)} \\ T_{-k} \\ T_{-(k+1)} \end{bmatrix}.$$

□

To find  $T_{-k}$ , we may refer to [1] that  $T_{-k} = \begin{vmatrix} T_{k-1} & T_k \\ T_{k-2} & T_{k-1} \end{vmatrix}$ . Thus for instance,  $T_{-8} = \begin{vmatrix} T_7 & T_8 \\ T_6 & T_7 \end{vmatrix} = 4$  and  $T_{-9} = \begin{vmatrix} T_8 & T_9 \\ T_7 & T_8 \end{vmatrix} = -8$ . Similarly  $T_{-10} = 5$ . So we have  $a_{-9} = [3 \ -2 \ -1] \begin{bmatrix} T_{-8} \\ T_{-9} \\ T_{-10} \end{bmatrix} = 23$ , while  $a_9 = [3 \ -2 \ -1] \begin{bmatrix} T_{10} \\ T_9 \\ T_8 \end{bmatrix} = 241$ . Therefore, it follows, for example,  $T_{30} = a_9 T_{21} - a_{-9} T_{12} + T_3 = 29, 249, 425$ .

For each  $n \in \mathbb{Z}$ , we define two sequences

$$P_n = T_n + T_{-n} \quad \text{and} \quad Q_n = T_n - T_{-n}.$$

Then it is easy to have the table

$n   \dots$	1	2	3	4	5	6	7	8	9	10	11	12 $\dots$
$T_n   \dots$	1	1	2	4	7	13	24	44	81	149	274	504 $\dots$
$T_{-n}   \dots$	0	1	-1	0	2	-3	1	4	-8	5	7	-20 $\dots$
$P_n   \dots$	1	2	1	4	9	10	25	48	<b>73</b>	154	281	<b>484</b> $\dots$
$Q_n   \dots$	1	0	3	4	5	16	23	40	89	<b>144</b>	<b>267</b>	524 $\dots$

From the table, we notice  $3 + 4 + 9 = 16$  and  $73 + 144 + 267 = 484$ , etc.

**THEOREM 3.3.** *Let  $k \in \mathbb{Z}$ . Then the sequences  $\{P_k\}$  and  $\{Q_k\}$  satisfy inter-recurrence relations  $P_{k-3} + Q_{k-2} + Q_{k-1} = P_k$  and  $Q_{k-3} + P_{k-2} + P_{k-1} = Q_k$  with initials  $\{1, 2, 1\}$  for  $P_k$  and  $\{1, 0, 3\}$  for  $Q_k$ . Furthermore*

- (1)  $\frac{1}{2}(P_k + Q_k) = T_k$  and  $\frac{1}{2}(P_k - Q_k) = T_{-k}$
- (2)  $a_k + a_{-k} = [3 \ -2 \ -1] \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \end{bmatrix} + 4(T_k + T_{k-2})$

*Proof.* It is easy to see that

$$\begin{aligned} P_k &= T_k + T_{-k} = (T_{k-3} + T_{k-2} + T_{k-1}) + (T_{-k+3} - T_{-k+2} - T_{-k+1}) \\ &= (T_{k-3} + T_{-(k-3)}) + (T_{k-2} - T_{-(k-2)}) + (T_{k-1} - T_{-(k-1)}) \\ &= P_{k-3} + Q_{k-2} + Q_{k-1}, \end{aligned}$$

and analogously  $Q_{k-3} + P_{k-2} + P_{k-1} = Q_k$ . Hence (1) is easy to see.

Now we note that

$$\begin{aligned} T_{k+1} + T_{-(k-1)} &= (T_{k-2} + T_{k-1} + T_k) + T_{-(k-1)} \\ &= (T_{k-2} + T_k) + (T_{k-1} + T_{-(k-1)}) = T_{k-2} + T_k + P_{k-1}, \end{aligned}$$

and similarly

$$T_{k-1} + T_{-(k+1)} = (T_{k+1} - T_{k-2} - T_k) + T_{-(k+1)} = -(T_{k-2} + T_k) + P_{k+1}.$$

Therefore together with Theorem 3.2, it follows that

$$\begin{aligned} a_k + a_{-k} &= [3 - 2 - 1] \begin{bmatrix} T_{k+1} + T_{-(k-1)} \\ T_k + T_{-k} \\ T_{k-1} + T_{-(k+1)} \end{bmatrix} \\ &= [3 - 2 - 1] \begin{bmatrix} T_{k-2} + T_k + P_{k-1} \\ P_k \\ -(T_{k-2} + T_k) + P_{k+1} \end{bmatrix} \\ &= [3 - 2 - 1] \left( \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \end{bmatrix} + \begin{bmatrix} T_{k-2} + T_k \\ 0 \\ -(T_{k-2} + T_k) \end{bmatrix} \right) \\ &= [3 - 2 - 1] \left( \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \end{bmatrix} + (T_k + T_{k-2}) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \\ &= [3 - 2 - 1] \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \end{bmatrix} + 4(T_k + T_{k-2}). \end{aligned}$$

□

Owing to Theorem 3.3 we first construct the table of  $P_n$  and  $Q_n$  recursively, and then obtain tribonacci numbers from the table. Indeed,

$k   \dots$	10	11	12	13	14	15	16	17	18 $\dots$
$P_k   \dots$	154	281	484	945	1714	3089	5824	10609	19410
$Q_k   \dots$	7144	267	524	909	1696	3183	5712	10609	
$\frac{1}{2}(P_k + Q_k)   \dots$					1705	3136	5768	10609	
$\frac{1}{2}(P_k - Q_k)   \dots$					9	-47	56	0	

shows  $T_{17} = \frac{1}{2}(P_{17} + Q_{17}) = 10609$  and  $T_{-17} = \frac{1}{2}(P_{17} - Q_{17}) = 0$ . Hence

$$\begin{aligned} a_{17} + a_{-17} &= [3 - 2 - 1] \begin{bmatrix} P_{16} \\ P_{17} \\ P_{18} \end{bmatrix} + 4(T_{15} + T_{17}) \\ &= [3 - 2 - 1] \begin{bmatrix} 5824 \\ 10609 \\ 19410 \end{bmatrix} + 4(3136 + 10609) = 31824 \end{aligned}$$

where, it can be compared with  $a_{17} = 31553$  and  $a_{-17} = -b_{17} = 271$ .



**THEOREM 3.4.** *Let  $n = kt + r$  with  $1 \leq r \leq k < n$ . Let  $(a_k, b_k)$  be the coefficients satisfying  $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$ .*

- (1)  $T_n$  is a linear combination of any three consecutive entries of  $r$ th column in the  $k$  columns tribonacci table.
- (2)  $T_n$  is expressed by the first three terms  $T_{2k+r}$ ,  $T_{k+r}$  and  $T_r$  of  $r$ th column.
- (3) For  $0 < i < t$ , if  $T_n = uT_{k(i+2)+r} + vT_{k(i+1)+r} + wT_{ki+r}$  with  $u, v, w \in \mathbb{Z}$  then  $T_n = (a_k u + v)T_{k(i+1)+r} + (b_k u + w)T_{ki+r} + uT_{k(i-1)+r}$ .

*Proof.*  $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$  in Theorem 2.3 yields

$$\begin{aligned} T_{kt+r} &= a_k T_{k(t-1)+r} + b_k T_{k(t-2)+r} + T_{k(t-3)+r} \\ &= a_k (a_k T_{k(t-2)+r} + b_k T_{k(t-3)+r} + T_{k(t-4)+r}) + b_k T_{k(t-2)+r} + T_{k(t-3)+r} \\ &= (a_k^2 + b_k) T_{k(t-2)+r} + (a_k b_k + 1) T_{k(t-3)+r} + a_k T_{k(t-4)+r} \\ &= (a_k (a_k^2 + b_k) + (a_k b_k + 1)) T_{k(t-3)+r} + (b_k (a_k^2 + b_k) + a_k) T_{k(t-4)+r} \\ &\quad + (a_k^2 + b_k) T_{k(t-5)+r}. \end{aligned}$$

Hence after  $i$  step ( $0 < i < t$ ), if we write

$$T_{kt+r} = uT_{k(i+2)+r} + vT_{k(i+1)+r} + wT_{ki+r}$$

with  $u, v, w \in \mathbb{Z}$  then in the next step we have

$$\begin{aligned} T_{kt+r} &= u(a_k T_{k(i+1)+r} + b_k T_{ki+r} + T_{k(i-1)+r}) + vT_{k(i+1)+r} + wT_{ki+r} \\ &= (a_k u + v)T_{k(i+1)+r} + (b_k u + w)T_{ki+r} + uT_{k(i-1)+r}. \end{aligned}$$

Continue this process to reach  $i = 1$ , then it follows that  $T_n$  is a linear combination of  $T_{2k+r}$ ,  $T_{k+r}$  and  $T_r$ .  $\square$

For example, for  $T_{56}$  we may take any  $k < 56$ , say  $k = 10$ . Since  $(a_{10}, b_{10}) = (443, 41)$  by Theorem 2.2,  $T_{56}$  can be obtained easily by Theorem 3.4 that

$$\begin{aligned} T_{56} &= 443T_{46} + 41T_{36} + T_{26} \\ &= (443^2 + 41)T_{36} + ((41)443 + 1)T_{26} + 443T_{16} \\ &= ((443)196290 + 18164)T_{26} + ((41)196290 + 443)T_{16} + 196290T_6 \\ &= (86974634)2555757 + (8048333)5768 + (196290)13 \\ &= 222, 332, 455, 004, 452, \end{aligned}$$

which is a 15 digit number. However, since  $T_n$  is composed of  $T_{n-k}$ ,  $T_{n-2k}$  and  $T_{n-3k}$ , it may be better to choose  $k \approx \frac{n}{3}$ . Indeed if we take  $\frac{56}{3} \approx 18 = k$ , then

$$T_{56} = a_{18}T_{38} - a_{-18}T_{20} + T_2$$

and the last term  $T_2 = 1$  is known easily.

LEMMA 3.5. *Let  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Then  $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ ,  $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$ , and  $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor & \text{if } x \in \mathbb{N} \\ -\lfloor x \rfloor - 1 & \text{if } x \notin \mathbb{N}. \end{cases}$*

THEOREM 3.6. *Assume the same context  $(a_k, b_k)$  as before. Then*

$$T_n = \begin{cases} a_{\frac{n}{3}} T_{\frac{2n}{3}} + b_{\frac{n}{3}} T_{\frac{n}{3}}, & \text{if } 3|n \\ a_{\lfloor \frac{n}{3} \rfloor} T_{\lfloor \frac{2n}{3} \rfloor + 1} + b_{\lfloor \frac{n}{3} \rfloor} T_{\lfloor \frac{n}{3} + \frac{1}{2} \rfloor + 1} + 1, & \text{if } 3 \nmid n. \end{cases}$$

*Proof.* From Theorem 3.1,  $T_n = a_k T_{n-k} + b_k T_{n-2k} + T_{n-3k}$  with  $a_k = 3T_k - T_{k-6}$  and  $b_k = -a_{-k}$ . Now let  $k = \lfloor \frac{n}{3} \rfloor$ . If  $3|n$  then  $k = \frac{n}{3}$ . Since  $T_0 = 0$ , we have

$$T_n = a_{\frac{n}{3}} T_{\frac{2n}{3}} + b_{\frac{n}{3}} T_{\frac{n}{3}}.$$

If  $3 \nmid n$ , then due to Lemma 3.5 we have

$$\begin{aligned} n - k &= n + (-\lfloor \frac{n}{3} \rfloor) = n + (\lfloor -\frac{n}{3} \rfloor + 1) = \lfloor n - \frac{n}{3} \rfloor + 1 = \lfloor \frac{2n}{3} \rfloor + 1, \\ n - 2k &= \lfloor \frac{2n}{3} \rfloor + 1 - \lfloor \frac{n}{3} \rfloor = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{3} + \frac{1}{2} \rfloor + 1 - \lfloor \frac{n}{3} \rfloor = \lfloor \frac{n}{3} + \frac{1}{2} \rfloor + 1. \end{aligned}$$

Moreover since  $n - 3k = 1$  or  $2$  and  $T_1 = T_2 = 1$ , it follows that

$$T_n = a_{\lfloor \frac{n}{3} \rfloor} T_{\lfloor \frac{2n}{3} \rfloor + 1} + b_{\lfloor \frac{n}{3} \rfloor} T_{\lfloor \frac{n}{3} + \frac{1}{2} \rfloor + 1} + 1.$$

□

For example,  $T_{67} = a_{22} T_{45} + b_{22} T_{23} + 1 = (664183)272809183135 + (1189)410744 + 1 = 181, 195, 222, 170, 528, 322$ , an 18 digit number.

#### 4. Sum of Tribonacci numbers in a row

THEOREM 4.1.  $\sum_{i=0}^t T_{4i} = \frac{1}{T_4^2} (T_{4t+4} + 6T_{4t} + T_{4t-4} - T_4)$ .

*Proof.* Let  $t = 3$ . Lemma 2.1 shows  $T_{4(4)} = 11T_{4(3)} + 5T_{4(2)} + T_4$ , so we have

$$\begin{aligned} T_4^2 \sum_{i=0}^3 T_{4i} &= 16T_{4(3)} + 16T_{4(2)} + 16T_4 + 16T_0 \\ &= (11T_{4(3)} + 5T_{4(2)} + T_4) + 5T_{4(3)} + 11T_{4(2)} + 15T_4 \\ &= T_{4(4)} + 5T_{4(3)} + 11T_{4(2)} + 15T_4 \\ &= T_{4(4)} + 5T_{4(3)} + (T_{4(3)} - 5T_4) + 15T_4 = T_{4(4)} + 6T_{4(3)} + 10T_4 \\ &= T_{4(4)} + 6T_{4(3)} + T_{4(2)} - T_4 = T_{4(3)+4} + 6T_{4(3)} + T_{4(3)-4} - T_4, \end{aligned}$$

since  $10T_4 = 40 = T_8 - T_4$ . Assume  $T_4^2 \sum_{i=0}^t T_{4i} = T_{4t+4} + 6T_{4t} + T_{4t-4} - T_4$  is true. Then it follows that

$$\begin{aligned} & T_{4(t+1)+4} + 6T_{4(t+1)} + T_{4(t+1)-4} - T_4 \\ &= [11T_{4t+4} + 5T_{4(t-1)+4} + T_{4(t-2)+4}] + 6T_{4(t+1)} + T_{4(t+1)-4} - T_4 \\ &= 16T_{4t+4} + (T_{4t+4} + 6T_{4t} + T_{4t-4} - T_4) \\ &= T_4^2 T_{4t+4} + T_4^2 \sum_{i=0}^t T_{4i} = T_4^2 \sum_{i=0}^{t+1} T_{4i}. \end{aligned}$$

□

Theorem 4.1 was discussed in [4] as a sum of  $4t$  subscripted tribonacci numbers. But in our context, it can be explained as a sum of entries of 4th column in the 4 columns tribonacci table. We now shall study the sum of entries of any  $r$ th column in the 4 columns tribonacci table. Consider  $T_n = T_{4t+r}$  ( $t \geq 0$  and  $1 \leq r \leq 4$ ) as an entry placed at the  $(t + 1)$ th row and  $r$ th column in the table, and let

$$s_t^{(4,r)} = \sum_{i=0}^t T_{4i+r} = T_r + T_{4+r} + \dots + T_{4t+r}$$

be the partial sum of  $t + 1$  entries of  $r$ th column.

**THEOREM 4.2.** Consider  $s_t^{(4,r)}$  with  $1 \leq r \leq 4$ . Then for  $t \geq 3$ ,

- (1)  $s_t^{(4,r)} = \begin{cases} 11s_{t-1}^{(4,r)} + 5s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} - T_4 & \text{if } r = 1 \\ 11s_{t-1}^{(4,r)} + 5s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} + T_4 & \text{if } r \neq 1. \end{cases}$
- (2)  $s_t^{(4,r)} = \begin{cases} \frac{1}{T_4^2}(T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} + T_4) & \text{if } r = 1 \\ \frac{1}{T_4^2}(T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} - T_4) & \text{if } r \neq 1. \end{cases}$
- (3)  $s_t^{(4,r)} = 12s_{t-1}^{(4,r)} - 6s_{t-2}^{(4,r)} - 4s_{t-3}^{(4,r)} - s_{t-4}^{(4,r)}$ .

*Proof.* The 4 columns tribonacci table makes the table of  $s_t^{(4,r)}$  as follows.

4 columns table					$t$	$s_t^{(4,1)}$	$s_t^{(4,2)}$	$s_t^{(4,3)}$	$s_t^{(4,4)}$
1	1	2	4		0	1	1	2	4
7	13	24	44	and	1	8	14	26	48
81	149	274	504		2	89	163	300	552
927	1705	3136	5768		3	1016	1868	3436	6320
10609	19513	35890	...		4	11625	21381	39326	72332

When  $t = 3$ , we notice  $1016 = (11)89 + (5)8 + 1 - 4$ , and it can be written as

$$s_3^{(4,1)} = 11s_2^{(4,1)} + 5s_1^{(4,1)} + s_0^{(4,1)} - T_4.$$

Similar to this, we observe that

$$s_3^{(4,2)} = 11s_2^{(4,2)} + 5s_1^{(4,2)} + s_0^{(4,2)} + T_4,$$

$$s_3^{(4,3)} = 11s_2^{(4,3)} + 5s_1^{(4,3)} + s_0^{(4,3)} + T_4,$$

$$\text{and } s_3^{(4,4)} = 11s_2^{(4,4)} + 5s_1^{(4,4)} + s_0^{(4,4)} + T_4.$$

Assume that  $s_{t-1}^{(4,r)} = 11s_{t-2}^{(4,r)} + 5s_{t-3}^{(4,r)} + s_{t-4}^{(4,r)} \pm T_4$  with minus sign if  $r = 1$ , otherwise plus. Then Theorem 2.2 together with induction hypothesis yields

$$\begin{aligned} s_t^{(4,r)} &= \sum_{i=0}^t T_{4i+r} = s_{t-1}^{(4,r)} + T_{4t+r} \\ &= (11s_{t-2}^{(4,r)} + 5s_{t-3}^{(4,r)} + s_{t-4}^{(4,r)} \pm T_4) + (11T_{4(t-1)+r} + 5T_{4(t-2)+r} + T_{4(t-3)+r}) \\ &= 11s_{t-1}^{(4,r)} + 5s_{t-2}^{(4,r)} + s_{t-3}^{(4,r)} \pm T_4, \end{aligned}$$

this proves (1). Moreover we can check the next identities that

$$T_{17} + 6T_{13} + T_9 + T_4 = 10609 + 6(927) + 81 + 4 = 4^2(1016) = T_4^2 s_3^{(4,1)},$$

$$T_{18} + 6T_{14} + T_{10} - T_4 = 19513 + 6(1705) + 149 - 4 = 4^2(1868) = T_4^2 s_3^{(4,2)},$$

$$T_{19} + 6T_{15} + T_{11} - T_4 = 35890 + 6(3136) + 274 - 4 = 4^2(3436) = T_4^2 s_3^{(4,3)},$$

$$T_{20} + 6T_{16} + T_{12} - T_4 = 66012 + 6(5768) + 504 - 4 = 4^2(6320) = T_4^2 s_3^{(4,4)}.$$

So if we assume  $T_4^2 s_{t-1}^{(4,r)} = T_{4t+r} + 6T_{4(t-1)+r} + T_{4(t-2)+r} \pm T_4$ , then

$$\begin{aligned} &T_{4(t+1)+r} + 6T_{4t+r} + T_{4(t-1)+r} \pm T_4 \\ &= (11T_{4t+r} + 5T_{4(t-1)+r} + T_{4(t-2)+r}) + 6T_{4t+r} + T_{4(t-1)+r} \pm T_4 \\ &= (T_{4t+r} + 6T_{4(t-1)+r} + T_{4(t-2)+r} \pm T_4) + 16T_{4t+r} \\ &= T_4^2 s_{t-1}^{(4,r)} + T_4^2 T_{4t+r} = T_4^2 (s_{t-1}^{(4,r)} + T_{4t+r}) = T_4^2 s_t^{(4,r)}, \end{aligned}$$

this is (2). On the other hand, it is noticed that

$$12s_3^{(4,1)} - 6s_2^{(4,1)} - 4s_1^{(4,1)} - s_0^{(4,1)} = 12(1016) - 6(89) - 4(8) - 1 = s_4^{(4,1)},$$

and so on. We assume  $s_{t-1}^{(4,r)} = 12s_{t-2}^{(4,r)} - 6s_{t-3}^{(4,r)} - 4s_{t-4}^{(4,r)} - s_{t-5}^{(4,r)}$ . Then

$$\begin{aligned} s_t^{(4,r)} &= s_{t-1}^{(4,r)} + T_{4t+r} \\ &= (12s_{t-2}^{(4,r)} - 6s_{t-3}^{(4,r)} - 4s_{t-4}^{(4,r)} - s_{t-5}^{(4,r)}) \\ &\quad + (11T_{4(t-1)+r} + 5T_{4(t-2)+r} + T_{4(t-3)+r}) \\ &= 12(s_{t-2}^{(4,r)} + T_{4(t-1)+r}) - T_{4(t-1)+r} - 6(s_{t-3}^{(4,r)} + T_{4(t-2)+r}) \\ &\quad + 11T_{4(t-2)+r} - 4(s_{t-4}^{(4,r)} + T_{4(t-3)+r}) + 5T_{4(t-3)+r} - s_{t-5}^{(4,r)} \end{aligned}$$

$$\begin{aligned}
&= 12s_{t-1}^{(4,r)} - 6s_{t-2}^{(4,r)} - 4s_{t-3}^{(4,r)} - s_{t-5}^{(4,r)} \\
&\quad - (T_{4(t-1)+r} - 11T_{4(t-2)+r} - 5T_{4(t-3)+r}) \\
&= 12s_{t-1}^{(4,r)} - 6s_{t-2}^{(4,r)} - 4s_{t-3}^{(4,r)} - s_{t-5}^{(4,r)} - T_{4(t-4)+r} \\
&= 12s_{t-1}^{(4,r)} - 6s_{t-2}^{(4,r)} - 4s_{t-3}^{(4,r)} - s_{t-4}^{(4,r)}.
\end{aligned}$$

□

Theorem 4.1 is a special case of (2) in Theorem 4.2.

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