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ON A CLOSED DEDUCTIVE SYSTEM OF A CS-ALGEBRA

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Abstract. It is known that the class of CI-algebras is a generalization of the class of BE-algebras [5]. Recently, K. H. Kim introduced the notion of a CS-algebra [4]. In this paper we discuss a closed deductive system of a CS-algebra, and we find some fundamental properties. Moreover, we study a CS-algebra homomorphism and a congruence relation.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([1, 2]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. H. S. Kim and Y. H. Kim defined a BE-algebra as a dualization of generalization of a BCK-algebra [3]. In [5], B. L. Meng introduced the notion of a CI-algebra as a generalization of a BE-algebra. In [4], K. H. Kim introduced the notion of a CS-algebra with two binary operations $*$ and $\cdot$. A CS-algebra is a CI-algebra under $*$, also having an associative $\cdot$ that is left and right distributive over $*$. In this paper, we introduce the concept of a closed deductive system of a CS-algebra and discuss some related properties. In Section 2, we recall some definitions for CI- and CS-algebras and their properties. Finally, we prove some theorems for a closed deductive system of a CS-algebra.

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2. Preliminaries

In this section we investigate some definitions and properties of a CI- and CS-algebras. The proofs of most properties in this section are omitted. Their proofs may be found in [4, 5].

DEFINITION 2.1. [5] Let X be a nonempty set, and let $*$ be a binary operation on X. Then $(X, *, 1)$ is said to be a CI-algebra if the following axioms hold:

(CI1) $x * x = 1$,
(CI2) $1 * x = x$,
(CI3) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

PROPOSITION 2.2. [5] *For any CI-algebra X, we have following properties:*

(1) $y * ((y * x) * x) = 1$,
(2) $(x * 1) * (y * 1) = (x * y) * 1$ *for all* $x, y \in X$.

Now, we review some notions which will be used later. A nonempty subset S of a CI-algebra X is said to be a *subalgebra* of X if $x * y \in S$ for any $x, y \in S$. A CI-algebra X is called *commutative* if

$$(x * y) * y = (y * x) * x \quad \text{for all } x, y \in X.$$

DEFINITION 2.3. [4] An algebra $(X, \cdot, *, 1)$ with two binary operations "$\cdot$" and "$*$" is said to be a CS-algebra if the following axioms are satisfied:

(CS1) $S(X) = (X, \cdot)$ is a semigroup,
(CS2) $C(X) = (X, *)$ is a CI- algebra,
(CS3) $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

For convenience, we denote the multiplication $x \cdot y$ by xy.

EXAMPLE 2.4. Let $X = \{1, a, b, c\}$ in which "$*$" and "$\cdot$" are defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	a	1	b
c	1	a	1	1

$\cdot$	1	a	b	c
1	1	1	1	1
a	1	a	b	c
b	1	b	1	1
c	1	c	1	1

Then it is easy to check that $(X, \cdot, *, 1)$ is a CS-algebra.

In [6], the author defined a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 1$, where $x, y \in X$.

Proposition 2.5. [4] *Let X be a CS-algebra. Then the following identities hold:*

(1) $x1 = 1$ *and* $1x = 1$ *for all* $x \in X$,
(2) $x \leq y$ *implies* $ax \leq ay$ *and* $xa \leq ya$ *for all* $x, y, a \in X$.

Definition 2.6. [4] A nonempty subset A of a CS-algebra X is said to be *left* (resp. *right*) *stable* if $xa \in A$ (resp. $ax \in A$) for all $x \in X$ and $a \in A$.

It follows from Proposition 2.5 and Definition 2.6 that every stable set contains the element 1.

Definition 2.7. [4] A nonempty subset F of a CS-algebra X is said to be a *left* (resp. *right*) *deductive system* if it satisfies the following axioms:

(DS1) F is a left (resp. right) stable subset of $S(X)$,
(DS2) For any $x, y \in C(X)$, $x * y \in F$ and $x \in F$ imply $y \in F$.

In a CS-algebra X, we have $x1 = 1x = 1$ for all $x \in X$. If F is a deductive system of X, then $1 = 1a \in F$ for any $a \in F$.

Example 2.8. Let $X = \{1, a, b, c\}$ in which "$*$" and "$\cdot$" are defined by

$*$	1	a	b	c
1	1	a	b	c
a	c	1	c	1
b	1	c	1	c
c	c	b	a	1

$\cdot$	1	a	b	c
1	1	1	1	1
a	1	a	b	c
b	1	b	b	1
c	1	c	1	c

Then X is a CS-algebra. It is easy to check that $F = \{1, c\}$ is a deductive system of $(X, \cdot, *, 1)$.

Definition 2.9. [4] Let X be a CS-algebra. A nonempty subset S of X is called a *subalgebra* of X if $x * y \in S$ and $xy \in S$ for all $x, y \in S$.

3. A closed deductive system of a CS-algebra

In this section we show some properties on a closed deductive system of a CS-algebra.

Definition 3.1. [4] A deductive system F of a CS-algebra X is said to be closed if $x \in F$ implies $x * 1 \in F$.

EXAMPLE 3.2. In Example 2.8, $F = \{1, c\}$ is a closed deductive system of X.

PROPOSITION 3.3. [4] *A deductive system of a CS-algebra X is closed if and only if it is a subalgebra of a CS-algebra X.*

THEOREM 3.4. *If X is a CS-algebra, then the following properties are equivalent:*

(1) $(\forall x \in X)\ A_x = \{y \in X \mid y \cdot x = 1\}$ *is a deductive system of X.*
(2) $(\forall a, b \in X)\ \ a \cdot b = 1$ *implies* $a \cdot z \cdot b = 1$ *for all* $z \in X$.

Proof. (1) $\Rightarrow$ (2). Let $a, b \in X$ such that $a \cdot b = 1$. Then $a \in A_b$. Since A_b is stable $a \cdot z \in A_b$ for every $z \in X$. Thus $a \cdot z \cdot b = 1$ for all $z \in X$.

(2) $\Rightarrow$ (1). Let $x, z \in X$ and $a \in A_x$. Then by hypothesis $a \cdot z \in A_x$. Therefore, $a \cdot z, z \cdot a \in A_x$. Let $a, z, x \in X$ be such that $a * z \in A_x$ and $a \in A_x$. Then

$$\begin{aligned} 1 &= (a * z) \cdot x \\ &= a \cdot x * z \cdot x \\ &= 1 * z \cdot x \qquad (\text{ since } a \in A_x) \\ &= z \cdot x, \end{aligned}$$

Hence, $z \in A_x$. Therefore, A_x is a deductive system of X. $\square$

THEOREM 3.5. *Let F be a left (resp. right) deductive system of a CS-algebra X. If there exists $b \in F$ such that $a = xb$* (resp. $a = bx$) *for all $a \in F$, then F is closed.*

Proof. Let F be a left (resp. right) deductive system of a CS-algebra X such that there exists $b \in F$ satisfying $a = xb$ (resp. $a = bx$) for all $a \in F$. Then, for any $a \in F$, $a * 1 = xb * 1 = xb * 1b = (x * 1)b \in F$ (resp. $a * 1 = bx * 1 = b(x * 1)$, since F is left (resp. right) stable. Thus, F is a closed left (resp. right)deductive system of X. $\square$

DEFINITION 3.6. [4] A *CS-algebra* X is said to be a *near CS-algebra* if $x * y = xy * y$ for all $x, y \in X$.

If X ia a near CS-algebra with 1, then 1 is the greatest element in X since $x * 1 = x1 * 1 = 1 * 1 = 1$ for all $x \in X$.

PROPOSITION 3.7. *Let X be a near CS-algebra. Define a set H by*

$$H = \{x \in X \mid x * 1 = 1\}.$$

Then H is a closed deductive system of X.

Proof. Let $a \in H$ and $x \in X$. Then by Definition 3.6 $ax * 1 = (ax)1 * 1 = 1 * 1 = 1$, which implies $ax \in H$. Similarly, $xa \in H$. Obviously, $1 \in H$. If $x * y \in H$ and $x \in H$, then $x * 1 = 1$ and $(x * y) * 1 = 1$. By Proposition 2.2 we have

$$y * 1 = 1 * (y * 1) = (x * 1) * (y * 1) = (x * y) * 1 = 1.$$

This shows that H is a deductive system of X. If $x \in H$, then $x * 1 = 1 \in H$. Therefore, H is a closed deductive system of X. $\square$

PROPOSITION 3.8. *Let X be a near CS-algebra and $a \in X$. Define $A(a)$ by*

$$A(a) = \{x \in X \mid a * x = 1\}.$$

Then $A(a)$ is a right stable subset of X.

Proof. Let $b \in A(a)$ and $x \in X$. Then $1 = a * b = ab * b$. Hence by Definition 3.6 we get $a * bx = a(bx) * bx = ((ab) * b)x = 1x = 1$, which implies $bx \in A(a)$. Therefore, $A(a)$ is a right stable subset of X. $\square$

Now, we study a CS-algebra homomorphism.

DEFINITION 3.9. [4] Let X and Y be CS-algebras. A mapping $f : X \to Y$ is called a *CS-algebra homomorphism* (briefly, *homomorphism*) if $f(x * y) = f(x) * f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$. In particular, the set $\{x \in X | f(x) = 1\}$ is called the kernel of f, and it is denoted by $kerf$.

PROPOSITION 3.10. *Let $f : X \to X$ be a CS-algebra homomorphism. Then $kerf$ is a closed deductive system of X.*

Proof. Let $x \in X$ and $a \in kerf$. Then we obtain $f(ax) = f(a)f(x) = 1 \cdot f(x) = 1$ and $f(xa) = f(x)f(a) = f(x) \cdot 1 = 1$. Hence $xa, ax \in kerf$. So $kerf$ is a stable subset of X. Also, let $x * y \in kerf$ and $x \in kerf$. Then $1 = f(x * y) = f(x) * f(y) = 1 * f(y) = f(y)$. Hence we have $y \in kerf$. Let $x \in kerf$. Then we get $f(x * 1) = f(x) * f(1) = 1 * 1 = 1$, which implies $x * 1 \in kerf$. Therefore, $kerf$ is a closed deductive system of X. $\square$

PROPOSITION 3.11. *Let $f : X \to X$ be a CS-algebra homomorphism. Then the following properties are satisfied:*

(1) *If $x \leq y$, then $f(x \cdot z) \leq f(y \cdot z)$ and $f(z \cdot x) \leq f(z \cdot y)$ for all $x, y, z \in X$.*
(2) *If F is a left (resp. right) deductive system of X, then $f(F)$ is a left (resp. right) deductive system of $f(X)$.*

Proof. (1) Let $x, y, z \in X$. If $x \le y$, then $f(x \cdot z) * f(y \cdot z) = (f(x) \cdot f(z)) * (f(y) \cdot f(z)) = (f(x) * f(y)) \cdot f(z) = f(x * y) \cdot f(z) = f(1) \cdot f(z) = 1 \cdot f(z) = 1$. This implies $f(x \cdot z) \le f(y \cdot z)$. Similarly, we have $f(z \cdot x) \le f(z \cdot y)$.

(2) Let $a' \in f(F)$ and $x' \in f(F)$. Then $f(a) = a'$ and $f(x) = x'$ for some $a, x \in F$. Since $ax, xa \in F$, we get $f(a) \cdot f(x) = f(a \cdot x) \in f(F)$ and $f(x) \cdot f(a) = f(x \cdot a) \in f(F)$, which imply that $f(F)$ is a stable subset of X. Now, suppose that $x' \in f(F)$, $y' \in f(X)$ and $x' * y' \in f(F)$. Then $x' = f(x)$, $y' = f(y)$ and $x' * y' = f(z)$ for some $x, z \in F$ and $y \in X$. Then $f(x) \in f(F)$ and $f(x) * f(y) = f(x * y) \in f(F)$. Since F is a deductive system of X, we get $y \in F$ and $y' = f(y) \in f(F)$. Therefore, $f(F)$ is a deductive system of $f(X)$. □

PROPOSITION 3.12. *Let $f : X \to Y$ be a CS-algebra homomorphism and $ker f = \{1\}$. If $f(x) \le f(y)$, then $x \le y$.*

Proof. If $f(x) \le f(y)$, then $f(x * y) = f(x) * f(y) = 1$. Hence $x * y$ is an element of $ker f$ and $x * y = 1$. Therefore $x \le y$. □

An element e of X is called a *unity* in a CS-algebra if $ex = xe = x$ for all $x \in X$.

PROPOSITION 3.13. *Let $f : X \to Y$ be a homomorphism. Then*

(1) *If X is commutative, then so is $f(X)$.*
(2) *If X is a near CS-algebra with unity e, then so is $f(X)$ with unity $f(e)$.*

Proof. Let $x', y', z' \in f(X)$. Then $f(x) = x', f(y) = y'$ and $f(z) = z'$ for some $x, y, z \in X$.

(1) Suppose that X is commutative. Then $(y' * x') * x' = (f(y) * f(x)) * f(x) = f(y * x) * f(x) = f((y * x) * x) = f((x * y) * y) = f(x * y) * f(y) = (f(x) * f(y)) * f(y) = (x' * y') * y'$. Therefore $f(X)$ is commutative.

(2) Suppose that X is a near CS-algebra with unity e . Then $x' * y' = f(x) * f(y) = f(x * y) = f(x \cdot y * y) = f(x \cdot y) * f(y) = (f(x) \cdot f(y)) * f(y) = x' \cdot y' * y'$. Let $f(e) = e'$. Then $e' \cdot x' = f(e) \cdot f(x) = f(e \cdot x) = f(x) = x' = f(x \cdot e) = f(x) \cdot f(e) = x' \cdot e'$. Therefore $f(X)$ is a near CS-algebra with unity $f(e)$. □

Let's consider a binary relation on X.

DEFINITION 3.14. [4] Let X be a CS-algebra and let ρ be a binary relation on X. Then we define following:

(1) ρ is said to be *right* (resp. *left*) *compatible* if $(x, y) \in \rho$ implies $(x * z, y * z) \in \rho$ (resp. $(z * x, z * y) \in \rho$) and $(x \cdot z, y \cdot z) \in \rho$ (resp. $(z \cdot x, z \cdot y) \in \rho$) for all $x, y, z \in X$,

(2) ρ is said to be *compatible* if $(x, y) \in \rho$ and $(u, v) \in \rho$ imply $(x * u, y * v) \in \rho$ and $(x \cdot u, y \cdot v) \in \rho$ for all $x, y, u, v \in X$,

(3) A compatible equivalence relation is called a *congruence relation.*

THEOREM 3.15. *Let X be a CS-algebra. Then an equivalence relation ρ on X is a congruence if and only if it is both left and right compatible.*

Proof. Assume that ρ is a congruence relation on X. Let $(x, y) \in \rho$. Since ρ is reflexive, we get $(z, z) \in \rho$ for all $z \in X$. It follows from the compatibility of ρ that $(x * z, y * z) \in \rho$ and $(x \cdot z, y \cdot z) \in \rho$. Hence ρ is right compatible. Similarly, ρ is left compatible.

Conversely, suppose that ρ is both left and right compatible. Let $(x, y) \in \rho$ and $(u, v) \in \rho$. Then $(x * u, y * u) \in \rho$ and $(x \cdot u, y \cdot u) \in \rho$ by the right compatibility. Using the left compatibility of ρ, we have $(y * u, y * v) \in \rho$ and $(y \cdot u, y \cdot v) \in \rho$. It follows from the transitivity of ρ that $(x * u, y * v) \in \rho$ and $(x \cdot u, y \cdot v) \in \rho$. Hence ρ is a congruence. □

Using the notion of left (resp. right) compatible relation, we give a characterization of a congruence relation.

For a binary relation ρ on a CS-algebra X, we denote

$$x\rho = \{y \in X \mid (x, y) \in \rho\} \text{ and } X/\rho = \{x\rho \mid x \in X\}.$$

THEOREM 3.16. *Let ρ be a congruence relation on a CS-algebra X. Then X/ρ is a CS-algebra under the operations*

$$x\rho * y\rho = (x * y)\rho \text{ and } (x\rho) \cdot (y\rho) = (x \cdot y)\rho$$

for all $x\rho, y\rho \in X/\rho$.

Proof. Since ρ is a congruence relation, the operations are well-defined. Clearly, $(X/\rho, *)$ is a CI-algebra and $(X/\rho, \cdot)$ is a semigroup. For every $x\rho, y\rho, z\rho \in X/\rho$, we have

$$\begin{aligned} x\rho \cdot (y\rho * z\rho) &= x\rho \cdot (y * z)\rho = (x \cdot (y * z))\rho \\ &= (x \cdot y * x \cdot z)\rho = (x \cdot y)\rho * (x \cdot z)\rho \\ &= (x\rho \cdot y\rho) * (x\rho \cdot z\rho), \end{aligned}$$

and

$$\begin{aligned} (x\rho * y\rho) \cdot z\rho &= (x * y)\rho \cdot z\rho = ((x * y) \cdot z)\rho \\ &= (x \cdot z * y \cdot z)\rho = (x \cdot z)\rho * (y \cdot z)\rho \\ &= (x\rho \cdot z\rho) * (y\rho \cdot z\rho). \end{aligned}$$

Thus X/ρ is a CS-algebra. □

THEOREM 3.17. *If ρ is a congruence relation on a CS-algebra X, then 1ρ is a closed deductive system of X.*

Proof. Let $a \in 1\rho$ and $x \in X$. Then $(1, a) \in \rho$. By Theorem 3.15 ρ is both left and right compatible. Note that $(1 \cdot x, a \cdot x) = (1, a \cdot x) \in \rho$ and $(x \cdot 1, x \cdot a) = (1, x \cdot a) \in \rho$. Hence $a \cdot x \in 1\rho$ and $x \cdot a \in 1\rho$.

Assume $x * y \in 1\rho$ and $x \in 1\rho$. Then $(1, x * y) \in \rho$ and $(1, x) \in \rho$. It follows that $(1*y, x*y) = (y, x*y) \in \rho$. Since ρ is reflexive and transitive, we get $(1, y) \in \rho$, which implies $y \in 1\rho$. Thus 1ρ is a deductive system of X.

If $x \in 1\rho$, then $(1, x) \in \rho$ and hence $(1 * 1, x * 1) = (1, x * 1) \in \rho$, that is, $x * 1 \in 1\rho$. Therefore, 1ρ is a closed deductive system of X. □

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