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SOME APPLICATION OF THE UNION OF TWO $\mathbb{k}$-CONFIGURATIONS IN $\mathbb{P}^2$

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Abstract. It has been proved that the union of two linear star-configurations in $\mathbb{P}^2$ of type s and t for either $3 \le t \le 10$ or $\binom{t}{2} - 1 \le s$ with $3 \le t$ has maximal Hilbert function. We extend the condition to $\left[\frac{1}{2}\binom{t}{2}\right] \le s$, so that it is true for either $3 \le t \le 10$ or $\left[\frac{1}{2}\binom{t}{2}\right] \le s$ with $3 \le t$, which extends the result of [6].

1. Introduction

Let $R = \mathbb{k}[x_0, x_1, \ldots, x_n] = \bigoplus_{i \ge 0} R_i$ be the standard graded polynomial ring in $(n+1)$-variables over an infinite field $\mathbb{k}$ and $A = R/I$ where I is a homogeneous ideal in R. Then $A = \bigoplus_{i \ge 0} A_i$ is also a graded ring. The Hilbert function of A is the function

$$\mathbf{H}(A, t) = \dim_{\mathbb{k}} R_t - \dim_{\mathbb{k}} I_t.$$

If $I := I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote the Hilbert function of $\mathbb{X}$ by

$$\mathbf{H}_{\mathbb{X}}(t) := \mathbf{H}(R/I_{\mathbb{X}}, t)$$

(see [2, 3]). Let $\mathbb{X}$ be a set of s points in $\mathbb{P}^2$. We say that $\mathbb{X}$ has *maximal Hilbert function* (for sets of s points) if

$$\mathbf{H}_{\mathbb{X}}(t) = \min\left\{s, \binom{t+2}{2}\right\}$$

for every $t \ge 0$.

Let $F_1, F_2, \ldots, F_r$ be general forms in $R = \mathbb{k}[x_0, x_1, \ldots, x_n]$ with $r \ge 3$. Then $\bigcap_{1 \le i < j \le r} (F_i, F_j) = (\tilde{F}_1, \ldots, \tilde{F}_r)$, where $\tilde{F}_i = \dfrac{\prod_{j=1}^{r} F_j}{F_i}$ for $i = 1, \ldots, r$ (see [1, Proposition 2.1]). The variety $\mathbb{X}$ in $\mathbb{P}^n$ of the ideal

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$\bigcap_{1\le i<j\le r}(F_i,F_j)=(\tilde{F}_1,\dots,\tilde{F}_r)$ is called a *star-configuration* in $\mathbb{P}^n$ of type r. Furthermore, if the F_i are all general linear forms in R, the star-configuration $\mathbb{X}$ is called a *linear star-configuration* in $\mathbb{P}^n$ of type r.

In this paper, we construct the specific union of two $\Bbbk$-configurations in $\mathbb{P}^2$ having maximal Hilbert function. As an application, we prove that if $\mathbb{X}$ is the union of two linear star-configurations in $\mathbb{P}^2$ of type s and t, then $\mathbb{X}$ has maximal Hilbert function for $3\le t$ and $\left[\frac{1}{2}\binom{t}{2}\right]\le s$, which generalizes the interesting result of [6].

2. The union of two $\Bbbk$-configurations in $\mathbb{P}^2$

In this section we will compute the Hilbert functions of some general unions of particular configurations of points in $\mathbb{P}^2$. We first recall some standard facts and definitions (see [2, 3]).

DEFINITION 2.1. A $\Bbbk$-configuration of points in $\mathbb{P}^2$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ which satisfy the following conditions: there exist integers $1\le d_1<\cdots<d_m$, and subsets $\mathbb{X}_1,\dots,\mathbb{X}_m$ of $\mathbb{X}$, and distinct lines $\mathbb{L}_1,\dots,\mathbb{L}_m\subseteq\mathbb{P}^2$ such that

(a) $\mathbb{X}=\bigcup_{i=1}^m\mathbb{X}_i$,
(b) $|\mathbb{X}_i|=d_i$ and $\mathbb{X}_i\subset\mathbb{L}_i$ for each $i=1,\dots,m$, and
(c) $\mathbb{L}_i$ $(1<i\le m)$ does not contain any points of $\mathbb{X}_j$ for all $j<i$.

In this case, the $\Bbbk$-configuration in $\mathbb{P}^2$ is said to be of type $(d_1,\dots,d_m)$.

REMARK 2.2. Any two $\Bbbk$-configurations in $\mathbb{P}^2$ of the same type have the same minimal free resolution, and so the same Hilbert function ([2, 3]). We recall that if $\mathbb{X}$ is a linear star-configuration in $\mathbb{P}^2$ of type r with $3\le r$, then $\mathbb{X}$ is a $\Bbbk$-configuration in $\mathbb{P}^2$ of type $\mathcal{T}=(1,2,\dots,r-1)$ (see [2, 3] for the definition of a (standard) $\Bbbk$-configuration in $\mathbb{P}^n$).

The following lemma is immediate from the definition of a $\Bbbk$-configuration in $\mathbb{P}^n$, and so we omit the proof.

LEMMA 2.3. *Let $\mathbb{X}$ be a $\Bbbk$-configuration in $\mathbb{P}^2$ of type $\mathcal{T}=(1,2,3,\dots,d-1,d+1,d+2,\dots,s)$ with $s\ge 3$. Then $\mathbb{X}$ has maximal Hilbert function*

$$\mathbf{H}_{\mathbb{X}}\ :\ 1\quad \binom{1+2}{2}\quad\cdots\quad\binom{2+(s-2)}{2}\quad\binom{2+(s-1)}{2}-d\quad\rightarrow .$$

We introduce the following example for the proof of Proposition 2.5.

EXAMPLE 2.4. Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star-configurations in $\mathbb{P}^2$ of type s and t defined by linear forms $L_1,\dots,L_s$ and $M_1,\dots,M_t$, respectively.

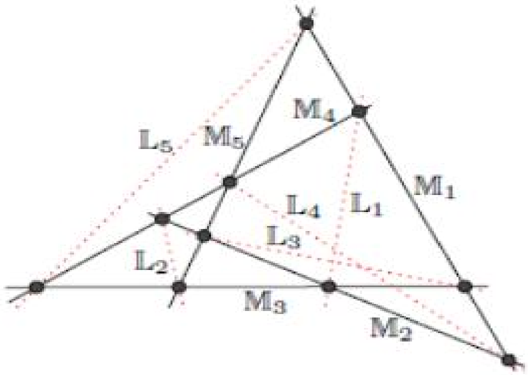

FIGURE 1. a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(2, 3, 4, 5, 6)$

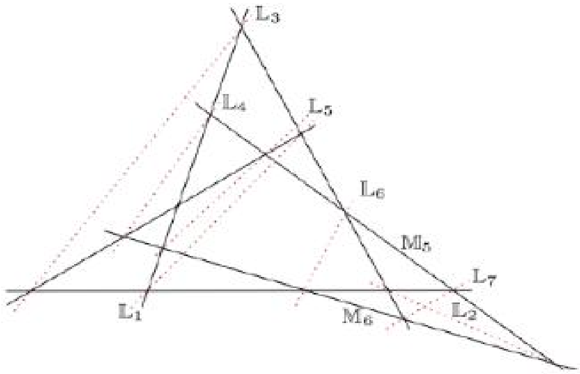

FIGURE 2. a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1, 2, \ldots, 8)$

(a) Let $t = 5$ and $s = 5$. As shown in Figure 1, $\mathbb{X} \cup \mathbb{Y}$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(2, 3, 4, 5, 6)$.
(b) Let $t = 6$ and $s = 7$. As shown in Figure 2, $\mathbb{X} \cup \mathbb{Y}$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1, 2, \ldots, 8)$.

PROPOSITION 2.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star-configurations in $\mathbb{P}^2$ of type s and t, respectively, with $3 \leq t \leq s$. If $s \geq \left[\frac{1}{2}\binom{t}{2}\right]$, then $\mathbb{X} \cup \mathbb{Y}$ has maximal Hilbert function.*

Proof. By Corollary 2.2 in [6], the result hold for $s \geq \binom{t}{2} - 1$. So we assume that $\left[\frac{1}{2}\binom{t}{2}\right] \leq s < \binom{t}{2} - 1$.

Let $\mathbb{X}$ and $\mathbb{Y}$ be defined by lines $\mathbb{L}_1, \ldots, \mathbb{L}_s$ and $\mathbb{M}_1, \ldots, \mathbb{M}_t$, respectively, where $\mathbb{L}_i$ and $\mathbb{M}_j$ are defined by linear forms L_i and M_j. By Theorems 3.1 and 3.2 in [4], the result holds for $t = 3$ and 4. So we assume that $t \geq 5$. Recall that $\mathbb{X}$ and $\mathbb{Y}$ are $\mathbb{k}$-configurations in $\mathbb{P}^2$ of type $(1, 2, \ldots, s-1)$ and $(1, 2, \ldots, t-1)$, respectively. For convenience and simplicity of the figure, we shall use Figure 3. First, we use the matrix

$$1 \quad 2 \quad \cdots \quad s-1$$

as a standard $\mathbb{k}$-configuration (a linear star-configuration $\mathbb{X}$) in $\mathbb{P}^2$ of type $(1, 2, \ldots, s-1)$ in Figure 3, i.e., we consider $\mathbb{X}$ a standard $\mathbb{k}$-configuration in $\mathbb{P}^2$. Second, we spread out the $\binom{t}{2}$-points of the other $\mathbb{k}$-configuration (linear star-configuration $\mathbb{Y}$) in $\mathbb{P}^2$ of type $(1, 2, \ldots, t-1)$ as points on a line, and make a partition as follows:

s-points lie on the line $\mathbb{N}_1$
the other $\alpha := \left(\binom{t}{2} - s\right)$-points lie on the line $\mathbb{N}_2$.

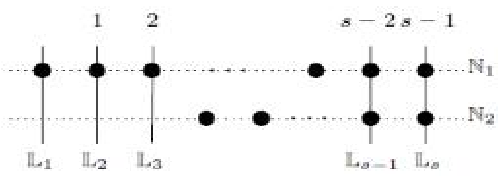

FIGURE 3. $(1, 2, \ldots, s-\alpha, s-\alpha+2, \ldots, s-1, s, s+1)$

Notice that it is always possible for every line $\mathbb{L}_i$ to pass through either a single point or two points in $\mathbb{Y}$ for $1 \leq i \leq s$. More precisely, we can prove this by induction on t, and it suffices to show when $s = \left[\frac{1}{2}\binom{t}{2}\right]$. By Example 2.4 (a) and (b), if $t = 5$ or $t = 6$, then it holds.

Now suppose $t > 6$. Consider a set $\mathbb{Z}$ of $2(t-2)$-points on two lines $\mathbb{M}_{t-1}$ and $\mathbb{M}_t$ except a single point $\wp$ defined by M_{t-1} and M_t. Then we make $(t-2)$-lines $\mathbb{L}_{s-t+3}, \ldots, \mathbb{L}_s$ pass through two distinct points in $\mathbb{Z}$ (see Example 2.4 (b)). Note that $\mathbb{W} := \mathbb{Y} - (\mathbb{Z} \cup \{\wp\})$ is a linear star-configuration in $\mathbb{P}^2$ of type $(t-2)$ defined by $M_1, \ldots, M_{t-2}$, and

$$\begin{array}{rcl} s-(t-2)-\left[\frac{1}{2}\binom{t-2}{2}\right] & = & \left[\frac{1}{2}\binom{t}{2}\right]-(t-2)-\left[\frac{1}{2}\binom{t-2}{2}\right] \\ & \geq & \frac{1}{2}(\binom{t}{2}-1)-(t-2)-\frac{1}{2}\binom{t-2}{2} \\ & = & 0. \end{array}$$

In other words, $s-(t-2)-\left[\frac{1}{2}\binom{t-2}{2}\right]=0$ or 1.

Case 1. Let $s-(t-2)-\left[\frac{1}{2}\binom{t-2}{2}\right]=0$. By induction on t, we can make $s-(t-2)$ lines $\mathbb{L}_1,\dots,\mathbb{L}_{s-t+2}$ pass through two distinct points in $\mathbb{W}$ (see Example 2.4 (a)). Hence $\mathbb{X}\cup(\mathbb{Y}-\{\wp\})$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(2,3,\dots,s+1)$, and so $\mathbb{X}\cup\mathbb{Y}$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1,2,3,\dots,s+1)$.

Case 2. Let $s-(t-2)-\left[\frac{1}{2}\binom{t-2}{2}\right]=1$. By induction on t, $\mathbb{L}_2,\dots,\mathbb{L}_{s-t+2}$ pass through two distinct points in $\mathbb{W}$, which implies that $\mathbb{X}\cup(\mathbb{Y}-\{\wp\})$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(3,4,\dots,s+1)$, and so $\mathbb{X}\cup\mathbb{Y}$ is a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1,3,4,\dots,s+1)$.

Therefore, it is from Cases 1, 2, and Lemma 2.3 that $\mathbb{X}\cup\mathbb{Y}$ has maximal Hilbert function, as we wished. □

If we couple the results in [4, 5] with Proposition 2.5, we obtain the following proposition.

PROPOSITION 2.6. *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star-configurations in $\mathbb{P}^2$ of type s and t, respectively, with $3\le t\le s$. If either $3\le t\le 10$ or $\left[\frac{1}{2}\binom{t}{2}\right]\le s$, then $\mathbb{X}\cup\mathbb{Y}$ has maximal Hilbert function.*

3. Additional comments and a question

First, the concept of a star-configuration in $\mathbb{P}^n$ has been developed to calculate the dimension of secant varieties to the variety of reducible forms (see [4]). We extend the definition of a star-configuration in $\mathbb{P}^n$, i.e., we call the variety $\mathbb{X}$ of the ideal

$$\bigcap_{1\le i_1<\cdots<i_r\le s}(F_{i_1},\dots,F_{i_r})$$

a star-configuration in $\mathbb{P}^n$ of type (r,s) with $2\le r$. In particular, if $r=n$, then $\mathbb{X}$ is a set of points in $\mathbb{P}^n$, which is call a *point star-configuration* in $\mathbb{P}^n$ of type s.

Hence we have a natural question of the union of two linear point star-configuration in $\mathbb{P}^n$.

QUESTION 3.1. Let $L_1,\dots,L_s$ and $M_1,\dots,M_t$ be general linear forms in $R=\mathbb{k}[x_0,x_1,\dots,x_n]$. Assume that $\mathbb{X}$ and $\mathbb{Y}$ are linear point star-configurations in $\mathbb{P}^n$ defined by L_i's and M_j's. Does $\mathbb{X}\cup\mathbb{Y}$ have maximal Hilbert function?

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