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# $q$-DEDEKIND-TYPE DAEHEE-CHANGHEE SUMS WITH WEIGHT $\alpha$ ASSOCIATED WITH MODIFIED $q$-EULER POLYNOMIALS WITH WEIGHT $\alpha$

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Abstract. Recently, $q$-Dedekind-type sums related to $q$-Euler polynomials was studied by Kim in [T. Kim, Note on $q$-Dedekind-type sums related to $q$-Euler polynomials, Glasgow Math. J. 54 (2012), 121-125]. It is aim of this paper to consider a $p$-adic continuous function for an odd prime to inside a $p$-adic $q$-analogue of the higher order Dedekind-type sums with weight related to modified $q$-Euler polynomials with weight by using Kim's $p$-adic $q$-integral.

## 1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively.

Let $v_p$ be normalized exponential valuation of $\mathbb{C}_p$ with

$$|p|_p = p^{-v_p(p)} = \frac{1}{p}.$$

When one speaks of $q$-extension, $q$ is variaously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1-q|_p < 1$ (see [1-16]).

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A $q$-extension of $p$-adic Haar measure is defined by Kim as follows: for any postive integer $N$,

$$\mu_q\left(a+p^N\mathbb{Z}_p\right)=(-q)^a\,\frac{(1+q)}{1+q^{p^N}}$$

for $0\leq a<p^N$ and this can be extended to a measure on $\mathbb{Z}_p$ (for details, see $[1-4,6-16]$).

The modified $q$-Euler polynomials with weight $\alpha$ are defined by Rim and Jeong as follows:

$$\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}(x)=\int_{\mathbb{Z}_p}q^{-y}\left(\frac{1-q^{\alpha(x+y)}}{1-q^{\alpha}}\right)d\mu_q(y) \tag{1.1}$$

for $n\in\mathbb{Z}_+:=\{0,1,2,3,...\}$. We note that

$$\lim_{q\to1}\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}(x)=E_n(x)$$

where $E_n$ are the famous Euler polynomials, which are defined by means of the following generating function:

$$\sum_{n=0}^{\infty}E_n(x)\,\frac{t^n}{n!}=e^{tx}\frac{2}{e^t+1},\ \ |t|<\pi$$

(for details, see [15]). Taking $x=0$ into (1.1), then, we have $\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}(0):=\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}$ are called modified $q$-Euler numbers with weight $\alpha$.

These numbers and polynomials have the following identities:

$$\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}=\frac{1+q}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\,\frac{1}{1+q^{\alpha l}}, \tag{1.2}$$

$$\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}(x)=\frac{1+q}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\,\frac{q^{\alpha lx}}{1+q^{\alpha l}}, \tag{1.3}$$

$$\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}(x)=\sum_{l=0}^{n}\binom{n}{l}q^{\alpha lx}\widetilde{\mathcal{E}}_{l,q}^{(\alpha)}\left(\frac{1-q^{\alpha x}}{1-q^{\alpha}}\right)^{n-l} \tag{1.4}$$

and

$$\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}(x)=\left(\frac{1-q^{\alpha d}}{1-q^{\alpha}}\right)\sum_{a=0}^{d-1}(-1)^a\,\widetilde{\mathcal{E}}_{n,q}^{(\alpha)}\left(\frac{x+a}{d}\right),\quad d\in\mathbb{N}\text{with } d\equiv1(mod\ \ 2) \tag{1.5}$$

(for more information, see [15]).

For any positive integer $h, k$ and $m$, Dedekind-type DC sums are defined by Kim in [6], [7] and [8] as follows:

$$S_m(h,k)=\sum_{M=1}^{k-1}(-1)^{M-1}\frac{M}{k}\overline{E}_m\left(\frac{hM}{k}\right)$$

where $\overline{E}_m(x)$ are the $m$-th periodic Euler function. Kim gave some interesting properties Dedekind-type DC sums. He also constructed a $p$-adic continuous function for an odd prime number to contain a $p$-adic $q$-analogue of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$ in [7]. After Simsek also studied to $q$-analogue of Dedekind-type sums. He also derived their interesting properties. By the same motivation, we, by using $p$-adic $q$-integral on $\mathbb{Z}_p$, will construct weighted $p$-adic $q$-analogue of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$.

## 2. Weighted $q$-analogue of Dedekind-type Sums associated with modified $q$-Euler polynomials with weight $\alpha$

Let $w$ denotes the Teichmüller character (mod $p$). For $x\in\mathbb{Z}_p^*:=\mathbb{Z}_p/p\mathbb{Z}_p$, set

$$\langle x:q\rangle = w^{-1}(x)\left(\frac{1-q^x}{1-q}\right).$$

Let $a$ and $N$ be positive integers with $(p,a)=1$ and $p\mid N$. We now consider the following

$$\widetilde{T}_q^{(\alpha)}\left(s,a,N:q^N\right)=w^{-1}(a)\langle x:q^{\alpha}\rangle^s\sum_{j=0}^{\infty}\binom{s}{j}q^{\alpha aj}\left(\frac{1-q^{\alpha N}}{1-q^{\alpha a}}\right)^j\widetilde{\mathcal{E}}_{j,q^N}^{(\alpha)}.$$

In particular, if $m+1\equiv 0(\mathrm{mod}\,p-1)$, then

$$\begin{aligned}&\widetilde{T}_q^{(\alpha)}\left(m,a,N:q^N\right)\\&=\left(\frac{1-q^{\alpha a}}{1-q^{\alpha}}\right)^m\sum_{j=0}^{m}\binom{m}{j}q^{\alpha aj}\widetilde{\mathcal{E}}_{j,q^N}^{(\alpha)}\left(\frac{1-q^{\alpha N}}{1-q^{\alpha a}}\right)^j\\&=\left(\frac{1-q^{\alpha N}}{1-q^{\alpha}}\right)^m\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha N\left(x+\frac{a}{N}\right)}}{1-q^{\alpha N}}\right)^m q^{-Nx}d\mu_{q^N}(x).\end{aligned}$$

That is, $\widetilde{T}_q^{(\alpha)}\left(m,a,N:q^N\right)$ is a continuous $p$-adic extension of $\left(\frac{1-q^{\alpha N}}{1-q^{\alpha}}\right)^n\widetilde{\mathcal{E}}_{n,q^N}^{(\alpha)}\left(\frac{a}{N}\right)$.

Let [.] be the Gauss' symbol and let $\{x\} = x - [x]$. Then, we consider $q$-analogue of the higher order Dedekind-type DC sums $\widetilde{S}_{m,q}^{(\alpha)}\left(h,k:q^{l}\right)$ as

$$\begin{aligned}
&\widetilde{S}_{m,q}^{(\alpha)}\left(h,k:q^{l}\right)\\
&=\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha k}}\right)\int_{\mathbb{Z}_p}q^{-lx}\left(\frac{1-q^{\alpha\left(lx+l\left\{\frac{hM}{k}\right\}\right)}}{1-q^{\alpha l}}\right)^{m}d\mu_{q^{l}}\left(x\right).
\end{aligned}$$

If $m+1\equiv 0\,(\mathrm{mod}p-1)$

$$\begin{aligned}
&\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha k}}\right)\\
&\qquad\qquad\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha k\left(x+\frac{hM}{k}\right)}}{1-q^{\alpha k}}\right)^{m}q^{-kx}d\mu_{q^{k}}\left(x\right)\\
&=\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right)^{m}\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m}\\
&\qquad\qquad\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha k\left(x+\frac{hM}{k}\right)}}{1-q^{\alpha k}}\right)^{m}q^{-kx}d\mu_{q^{k}}\left(x\right)
\end{aligned}$$

where $p\mid k$, $(hM,p)=1$ for each $M$. From (1.1), we note that

$$\begin{aligned}
&\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\widetilde{S}_{m,q}^{(\alpha)}\left(h,k:q^{k}\right)\\
&=\sum_{M=1}^{k-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right)\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m}(-1)^{M-1}\\
&\qquad\qquad\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha k\left(x+\frac{hM}{k}\right)}}{1-q^{\alpha k}}\right)^{m}q^{-kx}d\mu_{q^{k}}\left(x\right)\\
&=\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right)\widetilde{T}_{q}^{(\alpha)}\left(m,(hM)_{k}:q^{k}\right)
\end{aligned}\tag{2.1a}$$

where $(hM)_k$ denotes the integer $x$ such that $0 \leq x < n$ and $x \equiv \alpha \,(\mathrm{mod} k)$. It is not difficult to show that

(2.2)

$$\int_{\mathbb{Z}_p} q^{-t} \left( \frac{1 - q^{\alpha(x+t)}}{1 - q^{\alpha}} \right)^k d\mu_q(t)$$

$$= \left( \frac{1 - q^{\alpha m}}{1 - q^{\alpha}} \right)^k \frac{1+q}{1+q^m} \sum_{i=0}^{m-1} (-1)^i \int_{\mathbb{Z}_p} q^{-mt} \left( \frac{1 - q^{\alpha m \left(t + \frac{x+i}{m}\right)}}{1 - q^{\alpha m}} \right)^k d\mu_{q^m}(t).$$

By (2.1a) and (2.2), we easily see that

$$\left( \frac{1 - q^{\alpha N}}{1 - q^{\alpha}} \right) \int_{\mathbb{Z}_p} \left( \frac{1 - q^{\alpha N \left(x + \frac{a}{N}\right)}}{1 - q^{\alpha N}} \right)^m q^{-Nx} d\mu_{q^N}(x) \tag{2.3}$$

$$= \frac{1 + q^N}{1 + q^{Np}} \sum_{i=0}^{p-1} (-1)^i \left( \frac{1 - q^{\alpha N p}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left( \frac{1 - q^{\alpha p N \left(x + \frac{a + iN}{pN}\right)}}{1 - q^{\alpha p N}} \right) q^{-xpN} d\mu_{q^{pN}}(x)$$

From (2.1a), (2.2) and (2.3), we note that the $p$-adic integration is given by

$$\widetilde{T}_q^{(\alpha)}\left(s, a, N : q^N\right) = \frac{1 + q^N}{1 + q^{Np}} \sum_{\substack{0 \leq i \leq p-1 \\ a + iN \neq 0 (\mathrm{mod} p)}} (-1)^i \, \widetilde{T}_q^{(\alpha)} \left(s, (a + iN)_{pN}, p^N : q^{pN}\right)$$

such that

$$\widetilde{T}_q^{(\alpha)}\left(m, a, N : q^N\right) = \left( \frac{1 - q^{\alpha N}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left( \frac{1 - q^{\alpha N \left(x + \frac{a}{N}\right)}}{1 - q^{\alpha N}} \right)^m q^{-Nx} d\mu_{q^N}(x)$$

$$- \left( \frac{1 - q^{\alpha N p}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left( \frac{1 - q^{\alpha p N \left(x + \frac{a + iN}{pN}\right)}}{1 - q^{\alpha p N}} \right)^m q^{-pNx} d\mu_{q^{pN}}(x)$$

where $\left(p^{-1}a\right)_N$ denotes the integer $x$ with $0 \leq x < N$, $px \equiv a \,(\mathrm{mod} N)$ and $m$ is integer with $m + 1 \equiv 0 (\mathrm{mod} p - 1)$. Therefore, we procure the following

$$\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right)\widetilde{T}_{q}^{(\alpha)}\left(m,hM,k:q^{k}\right)$$
$$=\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\widetilde{S}_{m,q}^{(\alpha)}\left(h,k:q^{k}\right)$$
$$-\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\left(\frac{1-q^{\alpha kp}}{1-q^{\alpha k}}\right)\widetilde{S}_{m,q}^{(\alpha)}\left(\left(p^{-1}h\right),k:q^{pk}\right)$$

where $p\nmid k$ and $p\nmid hm$ for each $M$. Thus, we state the following definition.

Definition 2.1. Let $h,k$ be positive integer with $(h,k)=1$, $p\nmid k$. For $s\in\mathbb{Z}_p$, we define $p$-adic Dedekind-type DC sums as follows:

$$\widetilde{S}_{p,q}^{(\alpha)}\left(s:h,k:q^{k}\right)=\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right)\widetilde{T}_{q}^{(\alpha)}\left(m,hM,k:q^{k}\right).$$

Then, we can give the following theorem.

Theorem 2.2. *For $m+1\equiv 0(modp-1)$ and $\left(p^{-1}a\right)_N$ denotes the integer $x$ with $0\leq x<N$, $px\equiv a\,(modN)$, then, we have*

$$\widetilde{S}_{p,q}^{(\alpha)}\left(s:h,k:q^{k}\right)$$
$$=\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\widetilde{S}_{m,q}^{(\alpha)}\left(h,k:q^{k}\right)$$
$$-\left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\left(\frac{1-q^{\alpha kp}}{1-q^{\alpha k}}\right)\widetilde{S}_{m,q}^{(\alpha)}\left(\left(p^{-1}h\right),k:q^{pk}\right).$$

For $\alpha=1$, we have to Kim's results in [7]. This result seems to be interesting for further work in $[6-8,13]$.

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