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TOEPLITZ OPERATORS ON BLOCH-TYPE SPACES AND A GENERALIZATION OF BLOCH-TYPE SPACES

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Abstract. We deal with the boundedness of the n-th derivatives of Bloch-type functions and Toeplitz operators and give a relationship between Bloch-type spaces and ranges of Toeplitz operators. Also we prove that the vanishing property of $||uk_z^\alpha||_{s,\alpha}$ on the boundary of $\mathbb{D}$ implies the compactness of Toeplitz operators and introduce a generalization of Bloch-type spaces.

1. Introduction

Let dA denote the normalized area measure on the unit disk $\mathbb{D}$. For any real number α with $\alpha > -1$, we define $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA$ because $\int_{\mathbb{D}} (1-|z|^2)^\alpha dA(z) < \infty$ if and only if $\alpha > -1$. Since $\int_{\mathbb{D}} (1-|z|^2)^\alpha dA = \frac{1}{1+\alpha}$, dA_α is a probability measure on $\mathbb{D}$. For $p \geq 1$, the weighted Bergman space $L_a^p(dA_\alpha)$ consists of analytic functions on $\mathbb{D}$ which are also in $L^p(\mathbb{D}, dA_\alpha)$. Since $L_a^2(dA_\alpha)$ is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$, for each $z \in \mathbb{D}$, there is a function K_z^α in $L_a^2(dA_\alpha)$ such that $f(z) = < f, K_z^\alpha >$ for every f in $L_a^2(dA_\alpha)$, where $K_z^\alpha(w) = \dfrac{1}{(1-\overline{z}w)^{2+\alpha}}$ which is called the Bergman kernel and we define $k_z^\alpha(w) = \dfrac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}} = \dfrac{K_z^\alpha(w)}{||K_z^\alpha||_{2,\alpha}}$, where $||\cdot||_{2,\alpha}$ is the norm in the space $L^2(\mathbb{D}, dA_\alpha)$ and $< \cdot, \cdot >$ is the inner product in the space $L^2(\mathbb{D}, dA_\alpha)$.

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For a linear operator S on $L_a^2(dA_\alpha)$, S induces a function $\widetilde{S}$ on $\mathbb{D}$ given by $\widetilde{S}(z) =< Sk_z^\alpha, k_z^\alpha >$, $z \in \mathbb{D}$. The function $\widetilde{S}$ is called the Berezin transform of S.

For $u \in L^1(\mathbb{D}, dA_\alpha)$, the Toeplitz operator T_u^α with symbol u is the operator on $L_a^2(dA_\alpha)$ defined by $T_u^\alpha(f) = P_\alpha(uf)$, where P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto $L_a^2(dA_\alpha)$, in fact, $P_\alpha(f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+\alpha}} dA_\alpha(w)$.

For $\beta > 0$, the $\beta-$Bloch space B_β is the space of analytic fuctions f on $\mathbb{D}$ such that $||f||_\beta = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f'(z)| < \infty$ and $||\cdot||_\beta$ is a complete semi-norm on B_β. Moreover, B_β is a Banach space with norm of f equals to $||f|| = ||f||_\beta + |f(0)|$.

Also we define the little $\beta-$Bloch space B_β^0 to be the subspace of B_β consisting of the elements f such that $\lim_{|z|\to 1} (1-|z|^2)^\beta |f'(z)| = 0$. In fact, B_1 and B_1^0 are the classical Bloch space and little Bloch space, respectively.

Since P_α is the orthogonal projection, for any $f \in L^\infty$, T_f^α is bounded on the Bergman spaces $L_a^p(dA_\alpha)$, $p > 1$ because the Bergman projection P_α has norm 1 on L_a^2. Since L^∞ is dense in $L^1(\mathbb{D}, dA_\alpha)$, the Toeplitz operator T_u^α with symbol u in $L^1(\mathbb{D}, dA_\alpha)$ is densely defined on $L_a^2(dA_\alpha)$.

Many mathematicians working in operator theory are characterized the boundedness and compactness of Toeplitz operators. For references, see for example, [1], [2], [3].

In this paper, we study Toeplitz operators with special symbols on the $\beta-$Bloch spaces.

Section 2 of this paper contains properties of Bloch-type functions. Using the dominated property of $\beta-$Bloch-type functions, we investigate the boundedness of the $n-$th derivative of Toeplitz operators. We also prove that $T_u^\alpha : B_\beta \to B_{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}}$ is a compact linear operator under the vanishing property of u on the boundary and we get codomains of $D^{(n)}$, where $D^{(n)}$ is the $n-$th derivative operator. In Section 3, we introduce a generalization of Bloch-type spaces and we prove that the compactness of $T_u^\alpha : B_\beta \to B_{2+\frac{\alpha}{2}+\beta} - \frac{\alpha+2}{s'}$ is a special case of $T_u^\alpha : B_\beta \to E_{\frac{\alpha+2}{s'}-\beta}$.

Throughout the paper, we use p' to denote the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ and we use the symbol $A \preceq B$ ($A \approx B$, respectively) for nonnegative constants A and B to indicate that A is dominated by B times some positive constant ($A \preceq B$ and $B \preceq A$, respectively).

2. β-Bloch-type functions

For $\beta > 0$, the $\beta-$Bloch spaces B_β are Banach spaces with norm of f equals to $||f||_\beta + |f(0)|$ which coincides with the quotient norm on B_β/K where K is the closed subspace of constant functions. For $0 < \beta$, $(B^0_\beta)^* = L^1_a$ and $(L^1_a)^* = B_\beta$ (see [5]).

LEMMA 2.1. *Suppose $\beta > 1$ and $f \in B_\beta$. If $f(0) = 0$ then for any natural numbet n, $|f^{(n)}(z)| \preceq \dfrac{||f||_\beta}{(1-|z|^2)^{\beta+n-1}}$ for all $z \in \mathbb{D}$.*

Proof. Suppose $\beta < \beta'$. Since $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta'}|f'(z)| < +\infty$, f' is an analytic function in $L^1(\mathbb{D}, dA_{\beta'})$ and hence

$$f'(z) = \int_{\mathbb{D}} \frac{f'(w)}{(1-z\overline{w})^{2+\beta'}} dA_{\beta'}(w).$$

Taking the line integral from 0 to z, we get

$$\begin{aligned} f(z) &= \int_{\mathbb{D}} f'(w) \int_0^z \frac{dt}{(1-\overline{w}t)^{2+\beta'}} dA_{\beta'}(w) \\ &= \frac{1}{1+\beta'} \int_{\mathbb{D}} \frac{f'(w)}{\overline{w}} \Big(\frac{1}{(1-\overline{w}z)^{1+\beta'}} - 1 \Big) dA_{\beta'}(w) \\ &= \frac{1}{1+\beta'} \int_{\mathbb{D}} \frac{f'(w)}{\overline{w}(1-\overline{w}z)^{1+\beta'}} dA_{\beta'}(w). \end{aligned}$$

Here the 3rd equality comes from $\displaystyle\int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta'}}{\overline{w}} w^n dA(w) = 0$ and Taylor's series. Thus we get

$$|f(z)| \preceq \int_{\mathbb{D}} \frac{||f||_\beta (1-|w|^2)^{\beta'-\beta}}{|\overline{w}(1-z\overline{w})^{1+\beta'}|} dA(w)$$

and

$$|f'(z)| \preceq \int_{\mathbb{D}} \frac{||f||_\beta (1-|w|^2)^{\beta'-\beta}}{|1-z\overline{w}|^{2+\beta'}} dA(w).$$

Notice that for $\lambda > 0$, $\dfrac{1}{(1-z\overline{w})^{\lambda}} = \sum_{m=0}^{\infty} \dfrac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)} z^n \overline{w}^n$ and $\dfrac{\Gamma(m+\lambda)^2}{m!\Gamma(m+t)}$ $\approx m^{2\lambda-t-1}$ by Stirling's formula. Then we get

$$\begin{aligned} |f(z)| &\preceq \frac{||f||_\beta}{\Gamma(\frac{1}{2}+\frac{\beta'}{2})^2} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2}+\frac{\beta'}{2})^2}{\Gamma(m+1)^2} B(m+\frac{1}{2}, \beta'-\beta+1)|z|^{2m} \\ &\approx \frac{||f||_\beta}{(1-|z|^2)^{\beta-1}} \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\preceq \frac{||f||_\beta}{\Gamma(1+\frac{\beta'}{2})^2} \sum_{m=0}^{\infty} \frac{\Gamma(m+1+\frac{\beta'}{2})^2}{\Gamma(m+1)^2} B(m+1, \beta'-\beta+1)|z|^{2m} \\ &\approx \frac{||f||_\beta}{(1-|z|^2)^{\beta}}. \end{aligned}$$

Since $f''(z) = (2+\beta') \int_{\mathbb{D}} \dfrac{\overline{w} f'(w)}{(1-z\overline{w})^{3+\beta'}} (1-|w|^2)^{\beta'} dA(w)$,

$$\begin{aligned} |f''(z)| &\preceq \frac{||f||_\beta}{\Gamma(\frac{3}{2}+\frac{\beta'}{2})^2} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2}+\frac{\beta'}{2})^2}{\Gamma(m+1)^2} B(m+\frac{3}{2}, \beta'-\beta+1)|z|^{2m} \\ &\approx \frac{||f||_\beta}{(1-|z|^2)^{\beta+1}}. \end{aligned}$$

By the mathematical induction, $|f^{(n)}(z)| \preceq \dfrac{||f||_\beta}{(1-|z|^2)^{\beta+n-1}}$. □

THEOREM 2.2. *Suppose $\beta > 1$ and $f \in \overline{B_\beta} = \{f \in B_\beta : f(0) = 0\}$. Then $D^{(n)}(\overline{B_\beta}) \subset B_{\beta+n}$ for all natural number n, where $D^{(n)}$ denote the n-th derivative operator.*

Proof. It follows immediately from Lemma 2.1. □

The following lemma is Lemma 4.2.2 in [4].

LEMMA 2.3. *Suppose $\beta > -1$ and $t > 0$. Then* $\displaystyle\int_{\mathbb{D}} \frac{dA_\beta(w)}{|1-z\overline{w}|^{2+\beta+t}} \approx$ $(1-|z|^2)^{-t}$ *as* $|z| \to 1^-$.

By the reproducing property, for $g \in L^1_a$, $g(z) = \int_{\mathbb{D}} g(w)\overline{K^\beta_z(w)}dA_\beta(w)$. Suppose $f \in B_\beta$. Since $\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f'(z)| < \infty$, f' is in $L^1_a(\mathbb{D}, dA_\beta)$ and hence $f'(z) = \int_{\mathbb{D}} \frac{f'(w)}{(1-z\overline{w})^{2+\beta}} dA_\beta(w) = \int_{\mathbb{D}} f'(w)\overline{K^\beta_z(w)}dA_\beta(w)$. Taking the line integral from 0 to z, we get

$$\begin{aligned} f(z) - f(0) &= \int_{\mathbb{D}} f'(w) \int_0^z \frac{1}{(1-t\overline{w})^{2+\beta}} dt dA_\beta(w) \\ &= \frac{1}{1+\beta} \int_{\mathbb{D}} \frac{f'(w)}{\overline{w}} \Big(\frac{1}{(1-z\overline{w})^{1+\beta}} - 1 \Big) dA_\beta(w). \end{aligned}$$

Since $\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{\overline{w}} dA(w) = 0 = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{\overline{w}} w^n \ dA(w)$ for every natural number n, use Taylor series to obtain

$$\int_{\mathbb{D}} \frac{f'(w)}{\overline{w}} dA_\beta(w) = 0.$$

Thus $f(z) = \frac{1}{1+\beta} \int_{\mathbb{D}} \frac{f'(w)}{\overline{w}(1-z\overline{w})^{1+\beta}} dA_\beta(w) + f(0)$. In particular, for any natural number n, $\frac{1}{1+\beta} \int_{\mathbb{D}} \frac{w^{n-1}}{\overline{w}(1-z\overline{w})^{1+\beta}} dA_\beta(w) = \frac{1}{n} z^n$ and $f^{(n)}(z)$ $= (-1)^n \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+2)} \int_{\mathbb{D}} \frac{\overline{w}^{n-1} f'(w)}{(1-z\overline{w})^{n+\beta+1}} \ dA_\beta(w)$.

Notice that

$$\begin{aligned} \frac{1}{(1-z\overline{w})^{1+\beta}} &= \sum_{m=0}^\infty \binom{-1-\beta}{m} (-z\overline{w})^m \\ &= \sum_{m=0}^\infty \frac{\Gamma(\beta+m+1)}{m!\Gamma(\beta+1)} (z\overline{w})^m \end{aligned}$$

and Stirling's formula implies $\frac{\Gamma(a+x)}{\Gamma(b+x)} \approx x^{a-b}$.

Using a simple calculation to obtain Theorem 2.4 which is Proposition 7 in [5]. The calculation method gives a sharp index to codomains of Toeplitz operators (see Theorem 2.6).

Theorem 2.4.

(1) *Suppose* $\beta > 1$ *and* $f \in B_\beta$. *Then* $(1-|z|^2)^{\beta-1} f(z)$ *is bounded on* $\mathbb{D}$.

(2) *If there is a positive real number* β *such that* $(1-|z|^2)^{\beta} f(z)$ *is bounded on* $\mathbb{D}$ *then* $f \in B_{1+\beta}$ *and vice versa.*

Proof. (1) Suppose $\beta < \beta'$. Then $f(z) - f(0) = \int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta'} f'(w)}{\overline{w}(1-z\overline{w})^{1+\beta'}}$ $dA(w)$.

Since $(1-z\overline{w})^{-\frac{1}{2}-\frac{\beta'}{2}} = \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2}+\frac{\beta'}{2})}{m!\Gamma(\frac{1}{2}+\frac{\beta'}{2})} (z\overline{w})^m$,

$$\begin{aligned}
&|f(z)-f(0)| \\
&\le ||f||_\beta \int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta'-\beta}}{|\overline{w}(1-z\overline{w})^{1+\beta'}|} dA(w) \\
&= ||f||_\beta \sum_{m=0}^{\infty} \int_0^1 \frac{\Gamma(\frac{1}{2}+\frac{\beta'}{2}+m)^2}{\Gamma(\frac{1}{2}+\frac{\beta'}{2})^2 (m!)^2} r^{m-\frac{1}{2}} (1-r)^{\beta'-\beta} dr |z|^{2m} \\
&= \frac{||f||_\beta}{\Gamma(\frac{1}{2}+\frac{\beta'}{2})^2} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}+\frac{\beta'}{2}+m)^2}{\Gamma(m+1)^2} B\Big(m+\frac{1}{2}, \beta'-\beta+1\Big) |z|^{2m} \\
&= \frac{||f||_\beta}{\Gamma(\frac{1}{2}+\frac{\beta'}{2})^2} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}+\frac{\beta'}{2}+m)^2}{\Gamma(m+1)^2} \frac{\Gamma(m+\frac{1}{2})\Gamma(\beta'-\beta+1)}{\Gamma(m+\frac{1}{2}+\beta'-\beta+1)} |z|^{2m} \\
&\approx \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2}+\frac{\beta'}{2})^2 \Gamma(m+\frac{1}{2})}{\Gamma(m+1)^2 \Gamma(m+\frac{3}{2}+\beta'-\beta)} |z|^{2m} \\
&\approx \sum_{m=0}^{\infty} m^{1+\beta'+\frac{1}{2}-2-\frac{3}{2}-\beta'+\beta} |z|^{2m} = \sum_{m=0}^{\infty} m^{\beta-2} |z|^{2m} \\
&\approx \frac{1}{(1-|z|^2)^{\beta-1}}.
\end{aligned}$$

Here the last equivalence follows from $\beta - 1 > 0$.
Thus $(1-|z|^2)^{\beta-1} f(z)$ is bounded on $\mathbb{D}$.

(2) Suppose $(1-|z|^2)^{\beta} f(z)$ is bounded on $\mathbb{D}$ for some $\beta > 0$. If $\beta < \beta'$ then $(1-|z|^2)^{\beta'} f(z)$ is bounded on $\mathbb{D}$ and hence $f(z) = (1+\beta')$ $\int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta'} f(w)}{(1-z\overline{w})^{2+\beta'}} dA(w)$. Differentiating under the integral sign, we get

$$f'(z) = 1+\beta')(2+\beta')\int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta'} f(w)\overline{w}}{(1-z\overline{w})^{3+\beta'}} dA(w).$$

Since $\dfrac{1}{(1-z\overline{w})^{\frac{3}{2}+\frac{\beta'}{2}}} = \displaystyle\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2}+\frac{\beta'}{2})}{m!\Gamma(\frac{3}{2}+\frac{\beta'}{2})}(z\overline{w})^m,$

$$\begin{aligned} |f'(z)| &\preceq \int_{\mathbb{D}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2}+\frac{\beta'}{2})^2}{\Gamma(m+1)^2\Gamma(\frac{3}{2}+\frac{\beta'}{2})^2} |\overline{w}|(1-|w|^2)^{\beta'-\beta}|\overline{w}|^{2m}|z|^{2m} dA(w) \\ &\approx \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2}+\frac{\beta'}{2})^2}{\Gamma(m+1)^2} B\Big(m+\frac{3}{2}, \beta'-\beta+1\Big)|z|^{2m} \\ &\approx \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2}+\frac{\beta'}{2})^2}{\Gamma(m+1)^2} \frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+\frac{5}{2}+\beta'-\beta)}|z|^{2m} \\ &\approx \sum_{m=0}^{\infty} m^{3+\beta'+\frac{3}{2}-2-\frac{5}{2}-\beta'+\beta}|z|^{2m} = \sum_{m=0}^{\infty} m^{\beta}|z|^{2m} \\ &\approx \frac{1}{(1-|z|^2)^{\beta+1}}. \end{aligned}$$

Thus $(1-|z|^2)^{\beta+1}|f'(z)|$ is bounded on $\mathbb{D}$, that is, $f \in B_{1+\beta}$. □

LEMMA 2.5. *Suppose $f \in \overline{B_\beta}$. If $\beta > 0$ then $|f(w)| \leq \dfrac{||f||_\beta}{(1-|w|^2)^\beta}$ for all $w \in \mathbb{D}$, that is, the growth condition of f is dominated by $\dfrac{||f||_\beta}{(1-|w|^2)^\beta}$.*

Proof. Since $|f(w)| = |w\int_0^1 f'(wt)dt| \leq \Big|w\displaystyle\int_0^1 \frac{(1-|wt|^2)^\beta|f'(wt)|}{(1-|wt|^2)^\beta}dt\Big|$ $\leq \dfrac{||f||_\beta}{(1-|w|^2)^\beta}$, $|f(w)| \leq \dfrac{||f||_\beta}{(1-|w|^2)^\beta}$ for all $w \in \mathbb{D}$. □

Let $k_z^\alpha = \dfrac{K_z^\alpha}{||K_z^\alpha||}$ be the normalized repreducing kernel. Then for any analytic function f, $< f, K_z^\alpha >= f(z)$. Since $K_z^\alpha(w) = \dfrac{1}{(1-\overline{z}w)^{2+\alpha}}$, $(K_z^\alpha)'(w) = \dfrac{(2+\alpha)\overline{z}}{(1-\overline{z}w)^{3+\alpha}}$. Since $(1-|w|^2)^\beta\Big|\dfrac{z}{(1-\overline{z}w)^{3+\alpha}}\Big| \leq \dfrac{(1-|w|^2)^\beta}{(1-|w|)^{3+\alpha}}$ $= (1+|w|)^\beta(1-|w|)^{\beta-\alpha-3}$, for $\beta = 3+\alpha$, we get $K_z^\alpha \in B_\beta$, while for

$\beta > 3+\alpha$, $K_z^\alpha \in B_\beta^0$. Since $L^\infty \cap L_a^1(dA_\alpha)$ is dense in $L_a^1(dA_\alpha)$ and the dual space of B_β^0 is L_a^1, $k_z^\alpha \to 0$ weakly in B_β^0 as $z \to \partial\mathbb{D}$ because for any f in $L^\infty \cap L_a^1(dA_\alpha)$, $< f, k_z^\alpha > = (1-|z|^2)^{1+\frac{\alpha}{2}} f(z) \to 0$ as $z \to \partial\mathbb{D}$. Notice that for $u \in L^1(\mathbb{D}, dA_\alpha)$, $T_u^\alpha(f) = P_\alpha(uf)$, that is, $T_u^\alpha(f)(z) = \int_{\mathbb{D}} \frac{u(w)f(w)}{(1-z\overline{w})^{2+\alpha}} dA_\alpha(w)$ and there is a natural connection between the Bloch space and the Toeplitz operators via $P_1(L^\infty) = B_1$, where P_1 is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L_a^2(dA)$. For $u \in L^1(\mathbb{D}, dA_\alpha)$ and $f \in B_\beta$, we define $T_u^\alpha(f)(z) = \int_{\mathbb{D}} \frac{u(w)f(w)}{(1-z\overline{w})^{2+\alpha}} dA_\alpha(w)$. Let $WR(\alpha)$ $= \{u \in L^1(\mathbb{D}, dA_\alpha) : \sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} < +\infty$ for some $s \in (2,\infty)\}$.

Define $f(x) = \begin{cases} 2^{\frac{n}{3}} \ , \ \frac{1}{2^n} - \left(\frac{1}{2^{n+1}}\right)^2 \le x < \frac{1}{2^n} \\ 0 \ , \ \text{otherwise} \end{cases}$, where n is a natural number.

For each $z \in \mathbb{D}$, let $f(z) = f(|z|)$, that is, f is a radial function. If $0 \le \alpha$ then $\int_{\mathbb{D}} |f(w)| dA_\alpha(w) \le \int_{\mathbb{D}} |f(w)| dA(w) \le \sum_{n=1}^\infty \int_{\frac{1}{2^n} - (\frac{1}{2^{n+1}})^2}^{\frac{1}{2^n}} 2^{\frac{n}{3}} dr$ $\le \frac{1}{4}$. Thus f is in $L^1(\mathbb{D}, dA_\alpha)$. Since $\sup \left\{ \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{|1-\overline{z}w|^{2+\alpha}} : |w| < \frac{1}{2} \right.$ and $\left. z \in \mathbb{D} \right\} \le 2^{2+\alpha}$, $||fk_z^\alpha||_{3,\alpha}^3 = \int_{\mathbb{D}} |f(w)k_z^\alpha(w)|^3 dA_\alpha(w) \le 2^{(2+\alpha)3}$ $\sum_{n=1}^\infty \int_{\frac{1}{2^n} - (\frac{1}{2^{n+1}})^2}^{\frac{1}{2^n}} 2^n dr = 2^{(2+\alpha)3-2} = 2^{4+3\alpha} < \infty$. Thus $f \in WR(\alpha)$ and f is unbounded on $\mathbb{D}$, that is, $L^\infty(\mathbb{D})$ is a proper subset of $WR(\alpha)$. Since $||\overline{f}k_z^\alpha||_{s,\alpha} = ||fk_z^\alpha||_{s,\alpha}$ and for $t \in \mathbb{C}$, $||tfk_z^\alpha||_{s,\alpha} = |t| ||fk_z^\alpha||_{s,\alpha}$, $WR(\alpha)$ is closed under the formation of conjugations and a vector space.

Suppose $f \in WR(\alpha)$ and $z \in \mathbb{D}$. Since $\widetilde{|f|}(z) = \widetilde{T_{|f|}^\alpha}(z) = \int_{\mathbb{D}} |k_z^\alpha(w)|^2$ $|f(w)| dA_\alpha(w) \le ||fk_z^\alpha||_{2,\alpha} \le ||fk_z^\alpha||_{s,\alpha} \le ||f||_{WR(\alpha)}$, where $||f||_{WR(\alpha)}$ $= \sup_{z\in\mathbb{D}} ||fk_z^\alpha||_{s,\alpha}$, $\sup\{\widetilde{|f|}(z) : z \in \mathbb{D}\}$ is bounded and hence $|f| dA_\alpha$ is a Carleson measure on $\mathbb{D}$. Thus for each $u \in WR(\alpha)$, T_u^α is bounded on $L_a^p(dA_\alpha)$ for $1 < p < \infty$ and $||T_u^\alpha||_p \le C||f||_{WR(\alpha)}$ for some constant C, where $||T_u^\alpha||_p$ is the operator norm on $L_a^p(dA_\alpha)$ and $||f||_{WR(\alpha)}$ $= \sup_{z\in\mathbb{D}} ||fk_z^\alpha||_{s,\alpha}$ because P_α is bounded on $L_a^p(dA_\alpha)$. If $f \in L_a^2(dA_\alpha)$ then for any $w \in \mathbb{D}$, $(T_u^\alpha f)(w) = < T_u^\alpha f, K_w^\alpha > = < f, (T_u^\alpha)^* K_w^\alpha > =$

$\int_{\mathbb{D}} f(z)\overline{((T_u^\alpha)^* K_w^\alpha)(z)}dA_\alpha(z) = \int_{\mathbb{D}} f(z)(T_u^\alpha K_z^\alpha)(w)dA_\alpha(z)$. Thus T_u^α is the integral operator with kernel $T_u^\alpha K_z^\alpha(w)$ on $L_a^2(dA_\alpha)$.

Let's consider Toeplitz operators on the β-Bloch space. Suppose $u \in WR(\alpha)$, that is, $||u||_{WR(\alpha)} = \sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} < +\infty$ for some $s > 2$. For $f \in B_\beta$, $T_u^\alpha(f)(z) = \int_{\mathbb{D}} u(w)f(w)\overline{K_z^\alpha(w)}dA_\alpha(w)$, T_u^α is the integral operator with kernel $u(w)K_w^\alpha(z)$. Suppose $\beta > 0$ and $f \in \overline{B_\beta}$. Since $T_u^\alpha(f)(z) = \int_{\mathbb{D}} u(w)\overline{k_z^\alpha(w)}\frac{f(w)}{(1-|z|^2)^{1+\frac{\alpha}{2}}}dA_\alpha(w)$,

$$\begin{aligned}|T_u^\alpha(f)(z)| &\preceq \frac{||uk_z^\alpha||_{s,\alpha}}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\int_{\mathbb{D}} \frac{||f||_\beta^{s'}}{(1-|w|^2)^{\beta s'}}(1-|w|^2)^\alpha dA\Big)^{\frac{1}{s'}}\\ &= \frac{||uk_z^\alpha||_{s,\alpha}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\int_{\mathbb{D}}(1-|w|^2)^{\alpha-\beta s'}dA\Big)^{\frac{1}{s'}}.\end{aligned}$$

If $\alpha - \beta s' > -1$ then $\int_{\mathbb{D}}(1-|w|^2)^{\alpha-\beta s'}dA < +\infty$ and hence $T_u^\alpha(f)$ is well-defined. On the other hand, if $(1+\beta)s' - 2 - \alpha > 0$ then

$$\begin{aligned}&|T_u^\alpha(f)'(z)|\\ &= \Big|(2+\alpha)\int_{\mathbb{D}}\frac{\overline{w}u(w)f(w)}{(1-z\overline{w})^{3+\alpha}}dA_\alpha(w)\Big|\\ &\preceq \frac{||uk_z^\alpha||_{s,\alpha}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big|\int_{\mathbb{D}}\frac{|w|^{s'}(1-|w|^2)^{\alpha-\beta s'}}{(1-z\overline{w})^{s'}}dA\Big|^{\frac{1}{s'}}\\ &= \frac{||uk_z^\alpha||_{s,\alpha}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\sum_{m=0}^\infty \frac{\Gamma(m+\frac{s'}{2})^2 2}{\Gamma(m+1)^2\Gamma(\frac{s'}{2})}\int_0^1 r^{m+\frac{s'}{2}}(1-r)^{\alpha-\beta s'}dr|z|^{2m}\Big)^{\frac{1}{s'}}\\ &\approx \frac{||uk_z^\alpha||_{s,\alpha}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\sum_{m=0}^\infty m^{s'-2-\alpha+\beta s'-1}|z|^{2m}\Big)^{\frac{1}{s'}}\\ &\approx \frac{||uk_z^\alpha||_{s,\alpha}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\frac{1}{(1-|z|^2)^{(1+\beta)s'-2-\alpha}}\Big)^{\frac{1}{s'}} = \frac{||uk_z^\alpha||_{s,\alpha}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}+1+\beta-\frac{2+\alpha}{s'}}}.\end{aligned}$$

Thus $T_u^\alpha : B_\beta \to B_{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}}$ is a linear operator and $||T_u^\alpha|| \preceq ||u||_{WR}$. Since $1+\frac{\alpha}{2}-\frac{2+\alpha}{s'} < 0$, $T_u^\alpha : B_\beta \to B_{\beta+1}$ is also a bounded linear operator.

Moreover, $T_u^\alpha(f)^{(n)}(z) = \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+2)} \int_{\mathbb{D}} \frac{\overline{w}^n u(w) f(w)}{(1-z\overline{w})^{n+2+\alpha}} dA_\alpha(w)$. Since $k_z^\alpha(w) = \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$ and $|f(w)| \preceq \frac{||f||_\beta}{(1-|w|^2)^\beta}$,

$$
\begin{aligned}
&|T_u^\alpha(f)^{(n)}(z)| \\
&= \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+2)} \Big| \int_{\mathbb{D}} \frac{\overline{w}^n u(w) k_z^\alpha(w) f(w)}{(1-z\overline{w})^n (1-|z|^2)^{1+\frac{\alpha}{2}}} dA_\alpha(w) \Big| \\
&\le \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+2)} \frac{||u||_{WR}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}} \\
&\quad \times \Big(\int_{\mathbb{D}} \sum_{m=0}^\infty \frac{\Gamma(m+\frac{ns'}{2})^2}{\Gamma(m+1)^2 \Gamma(\frac{ns'}{2})^2} |w|^{2m+ns'} (1-|w|^2)^{\alpha-\beta s'} |z|^{2m} dA \Big)^{\frac{1}{s'}} \\
&= \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+2)} \frac{||u||_{WR}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}} \\
&\quad \times \Big(\sum_{m=0}^\infty \frac{2\Gamma(m+\frac{ns'}{2})^2}{\Gamma(m+1)^2 \Gamma(\frac{ns'}{2})^2} B(m+\frac{ns'}{2}+1, \alpha-\beta s'+1) |z|^{2m} \Big)^{\frac{1}{s'}} \\
&\approx \frac{||u||_{WR}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}} \Big(\sum_{m=0}^\infty m^{ns'-2-\alpha+\beta s'-1} |z|^{2m} \Big)^{\frac{1}{s'}} \\
&\approx \frac{||u||_{WR}||f||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}} \frac{1}{(1-|z|^2)^{n+\beta-\frac{2+\alpha}{s'}}}.
\end{aligned}
$$

Thus $T_u^\alpha(f)^{(n-1)} \in B_{1+\frac{\alpha}{2}+n+\beta-\frac{2+\alpha}{s'}}$. □

Summarizing the above observation, one has the following :

THEOREM 2.6. *Suppose $u \in WR(\alpha)$, that is, $\sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} < \infty$ for some $s > 2$ and $\beta > 0$. Then for each natural number n, $D^{(n-1)}(T_u^\alpha(B_\beta)) \subset B_{n+\beta+1+\frac{\alpha}{2}-\frac{2+\alpha}{s'}} \subset B_{n+\beta}$ and $T_u^\alpha : B_\beta \to B_{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}}$ is a bounded linear operator.*

COROLLARY 2.7. *Suppose $\beta > 0$ and $u \in WR(\alpha)$, where $\sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} < \infty$ for some $s > 2$. Then $T_u^\alpha : B_\beta \to B_{\beta+1}$ is a bounded linea operator.*

Proof. It is immedeately from the fact that $1 + \frac{\alpha}{2} - \frac{2+\alpha}{s'} < 0$. □

THEOREM 2.8. *Suppose $u \in WR(\alpha)$, that is, $\sup_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} < \infty$ for some $s > 2$ and $\alpha - \beta s' > -1$. If $||uk_z^\alpha||_{s,\alpha} \to 0$ as $z \to \partial\mathbb{D}$ then $T_u^\alpha : B_\beta \to B_{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}}$ is a compact linear operator.*

Proof. Let's show that T_u^α is compact on B_β. To do so, it is enough to show that if (f_n) is a bounded sequence in B_β and converges to 0 uniformly on compact subset of $\mathbb{D}$ then $||T_u^\alpha(f_n)||_{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}} \to 0$ as $n \to \infty$. Note that $(1-|z|^2)^{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}} |T_u^\alpha(f)'(z)| \preceq ||uk_z^\alpha||_{s,\alpha}||f||_\beta$. Let $M = \sup ||f_n||_\beta$. Take any $\varepsilon > 0$. Since $||uk_z^\alpha||_{s,\alpha} \to 0$ as $z \to \partial\mathbb{D}$, there is r such that $0 < r < 1$ and $\sup_{|z|>r} ||uk_z^\alpha||_{s,\alpha} < \frac{\varepsilon}{2M}$ and hence $\sup_{|z|>r} (1-|z|^2)^{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}} |T_u^\alpha(f_n)'(z)| \preceq \frac{\varepsilon}{2}$. Since $|f_n(w)| \le \frac{||f_n||_\beta}{(1-|w|^2)^\beta}$ and

$$\begin{aligned} T_u^\alpha(f_n)(z) &= (2+\alpha)\int_{\mathbb{D}} u(w)f_n(w)\frac{1}{(1-z\overline{w})^{2+\alpha}}dA_\alpha(w) \\ &= (2+\alpha)\int_{\mathbb{D}} u(w)k_z^\alpha(w)\frac{f_n(w)}{(1-|z|^2)^{1+\frac{\alpha}{2}}}dA_\alpha(w), \end{aligned}$$

$$\begin{aligned} |T_u^\alpha(f_n)(z)| &\preceq \frac{||uk_z^\alpha||_{s,\alpha}}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\int_{\mathbb{D}} \frac{||f_n||_\beta^{s'}}{(1-|w|^2)^{\beta s'}}(1-|w|^2)^\alpha dA(w)\Big)^{\frac{1}{s'}} \\ &= \frac{||uk_z^\alpha||_{s,\alpha}||f_n||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\int_{\mathbb{D}} (1-|w|^2)^{\alpha-\beta s'} dA(w)\Big)^{\frac{1}{s'}}. \end{aligned}$$

Since $\int_{\mathbb{D}} (1-|w|^2)^{\alpha-\beta s'} dA(w) < \infty$ and (f_n) converges to 0 uniformly on $\{z : |z| \le r\}$, $|T_u^\alpha(f_n)(0)| \to 0$ as $n \to \infty$. Since $|T_u^\alpha(f_n)'(z)| \preceq \frac{||u||_{WR(\alpha)}||f_n||_\beta}{(1-|z|^2)^{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}}}$ and (f_n) converges to 0 uniformly on $\{z : |z| \le r\}$, $\sup_{|z|\le r} (1-|z|^2)^{\beta+2+\frac{\alpha}{2}-\frac{2+\alpha}{s'}} |T_u^\alpha(f_n)'(z)| \to 0$ as $n \to \infty$.
Then we get $\lim_{n\to\infty} ||T_u^\alpha(f_n)|| = 0$. Thus T_u^α is a compact operator. □

3. A generalization of Bloch-type spaces

For $\alpha > -1$ and $z \in \mathbb{D}$, we define $U_z^\alpha f(w) = f \circ \varphi_z(w) \dfrac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$.
Since

$$\begin{aligned} U_z^\alpha U_z^\alpha f(w) &= U_z^\alpha f \circ \varphi_z(w) \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}} \\ &= f(w) \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}\varphi_z(w))^{2+\alpha}} \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}} = f(w), \end{aligned}$$

$(U_z^\alpha)^{-1} = U_z^\alpha$. For $f \in L_a^2$, $||U_z^\alpha f||_{2,\alpha}^2 = \int_{\mathbb{D}} |f \circ \varphi_z(\lambda)|^2 |\varphi_z^{'}(\lambda)|^{2+\alpha} dA_\alpha(\lambda)$ $= ||f||_{2,\alpha}$. Thus U_z^α is an isometry on $L_a^2(dA_\alpha)$. Hence U_z^α is a unitary operator on $L_a^2(dA_\alpha)$. Take any f in B_β. Then

$$\begin{aligned} &\sup_{w\in\mathbb{D}} (1-|w|^2)^\beta |(f\circ\varphi_z(w))'| \\ &= \sup_{w\in\mathbb{D}} (1-|\varphi_z(w)|^2)^\beta |f'(w)||\varphi_z^{'}(\varphi_z(w))| \\ &= \sup_{w\in\mathbb{D}} \Big(\frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{z}w|^2}\Big)^\beta |f'(w)| \frac{|1-\overline{z}w|^2}{1-|z|^2} \\ &= \sup (1-|w|^2)^\beta |f'(w)| (1-|z|^2)^{\beta-1} |1-\overline{z}w|^{2-2\beta} \end{aligned}$$

and hence $|| \cdot ||_1$ is Möbius invariant but the semi-norm is not Möbius invariant in the other case.

For a linear operator S on B_β, we define the conjugation operator S_z by $U_z^\alpha S U_z^\alpha$.

THEOREM 3.1. *For $u \in L^1(\mathbb{D}, dA_\alpha)$ and $z \in \mathbb{D}$, $(T_u^\alpha)_z = T_{u\circ\varphi_z}^\alpha$.*

Proof. Take any f in B_β and any w in $\mathbb{D}$. Since $(T_u^\alpha)_z = U_z^\alpha T_u U_z^\alpha$ and $(U_z^\alpha)^{-1} = U_z^\alpha$, it is enough to show that $U_z^\alpha T_u^\alpha = T_{u\circ\varphi_z}^\alpha U_z^\alpha$.
Since
$U_z^\alpha T_u^\alpha(w)$
$= T_u^\alpha(f)(\varphi_z(w)) \dfrac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}$

$$= (1+\alpha)\int_{\mathbb{D}} \frac{u(t)f(t)(1-|t|^2)^{\alpha}}{(1-\varphi_z(w)\bar{t})^{2+\alpha}} dA(t) \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{z}w)^{2+\alpha}}$$

$$= (1+\alpha)\int_{\mathbb{D}} u(t)f(t)(1-|t|^2)^{\alpha} \frac{(1-\bar{z}w)^{2+\alpha}}{(1-\bar{z}w-z\bar{t}+w\bar{t})^{2+\alpha}} dA(t) \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{z}w)^{2+\alpha}}$$

$$= (1+\alpha)\int_{\mathbb{D}} u(t)f(t)(1-|t|^2)^{\alpha} \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{z}w-z\bar{t}+w\bar{t})^{2+\alpha}} dA(t)$$

$$= (1+\alpha)\int_{\mathbb{D}} u\circ\varphi_z(s) f\circ\varphi_z(s)(1-|\varphi_z(s)|^2)^{\alpha}$$

$$\times \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}|\varphi_z'(s)|^2}{(1-\bar{z}w-z\overline{\varphi_z(s)}+w\overline{\varphi_z(s)})^{2+\alpha}} dA(s)$$

$$= (1+\alpha)\int_{\mathbb{D}} u\circ\varphi_z(s) f\circ\varphi_z(s) \frac{(1-|z|^2)^{\alpha}(1-|s|^2)^{\alpha}}{|1-\bar{z}s|^{2\alpha}}$$

$$\times \frac{(1-|z|^2)^{3+\frac{\alpha}{2}}}{|1-\bar{z}s|^4} \frac{(1-z\bar{s})^{2+\alpha}}{(1-|z|^2)^{2+\alpha}(1-w\bar{s})^{2+\alpha}} dA(s)$$

$$= (1+\alpha)\int_{\mathbb{D}} u\circ\varphi_z(s) f\circ\varphi_z(s) \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}(1-|s|^2)^{\alpha}}{(1-\bar{z}s)^{2+\alpha}(1-w\bar{s})^{2+\alpha}}$$

$$= \int_{\mathbb{D}} u\circ\varphi_z(s) U_z^{\alpha} f(s) \frac{dA_{\alpha}(s)}{(1-w\bar{s})^{2+\alpha}}$$

$$= T^{\alpha}_{u\circ\varphi_z}(U_z^{\alpha} f)(w), U_z^{\alpha} T_u^{\alpha} = T^{\alpha}_{u\circ\varphi_z} U_z^{\alpha}, thatis, (T_u^{\alpha})_z = T^{\alpha}_{u\circ\varphi_z}.$$

□

Let E_γ be the set of analytic functions f on $\mathbb{D}$ such that $||f||_{E_\gamma}$ is finite, where $||f||_{E_\gamma} = \sup\{(1-|z|^2)^{\gamma}||U_z^{\alpha} f||_{2+\frac{\alpha}{2}-\gamma} : z \in \mathbb{D}\}$. Then $||\cdot||_{E_\gamma}$ is a complete semi-norm on E_γ. Suppose that $\sup_{z\in\mathbb{D}} ||uk_z^{\alpha}||_{s,\alpha} < +\infty$ for some $s > 2$. Notice that $T_u^{\alpha}(B_\beta) \subset B_{2+\beta+\frac{\alpha}{2}-\frac{\alpha+2}{s'}} \subset B_{1+\beta}$. Consider $T_u^{\alpha} : B_\beta \to E_\gamma$, where $\gamma = \frac{\alpha+2}{s'} - \beta$. Put $t = 2 + \frac{\alpha}{2} - \gamma = 2 + \beta + \frac{\alpha}{2} - \frac{\alpha+2}{s'}$. If $g \in B_t$ and $3 + \alpha - 2t \geq 0$ then

$||U_z^{\alpha} g||_t$

$= \sup_{w\in\mathbb{D}} (1-|w|^2)^t |(U_z g)'(w)|$

$= \sup_{w\in\mathbb{D}} (1-|w|^2)^t \left|\left(g\circ\varphi_z(w) \frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\bar{z}w)^{2+\alpha}}\right)'\right|$

$$= \sup_{w\in\mathbb{D}} (1-|w|^2)^t \Big| g'(\varphi_z(w))\varphi'_z(w)\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}} + (2+\alpha)g(\varphi_z(w))\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{2+\alpha}}\Big|$$

$$= \sup_{w\in\mathbb{D}} \Big(\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}\Big)^t (1-|z|^2)^{1+\frac{\alpha}{2}} \times \Big| g'(w)\frac{(1-\overline{z}w)^2}{-1+|z|^2} + (2+\alpha)g(w)\frac{\overline{z}(1-\overline{z}w)^{3+\alpha}}{(1-|z|^2)^{3+\alpha}}\Big|$$

$$\preceq (1-|z|^2)^{1+\frac{\alpha}{2}+t} ||g||_t \sup_{w\in\mathbb{D}} \frac{1}{|1-\overline{z}w|^{2t}} \times \Big(\Big|\frac{|1-\overline{z}w|^{4+\alpha}}{(1-|z|^2)^{3+\alpha}} + (2+\alpha)\frac{|1-\overline{z}w|^{3+\alpha}}{(1-|z|^2)^{3+\alpha}}\Big|\Big)$$

$$= (1-|z|^2)^{t-2-\frac{\alpha}{2}} ||g||_t \sup_{w\in\mathbb{D}} |1-\overline{z}w|^{3+\alpha-2t}(|1-\overline{z}w| + 2+\alpha)$$

$$\leq (1-|z|^2)^{-\gamma} ||g||_t 2^{3+\alpha-2t}(2+2+\alpha).$$

Thus $(1-|z|^2)^{\gamma}||U_z^{\alpha}g||_t \preceq ||g||_t$, that is, $||g||_{E_\gamma} \preceq ||g||_t$. If $\gamma = 2+\frac{\alpha}{2}-\beta$ and $3+\alpha-2\beta \geq 0$ then $B_\beta \subset E_\gamma$. For $1 \leq 2+\frac{\alpha}{2}-\gamma$, $H^\infty \subset E_\gamma$. Moreover, $||f||_{E_\gamma} \preceq ||f||_\beta$ whenever $\gamma = 2+\frac{\alpha}{2}-\beta \geq \frac{1}{2}$. Since $U_z^{\alpha}1 = k_z^{\alpha}$,

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\gamma} ||U_z^{\alpha}1||_{2+\frac{\alpha}{2}-\gamma}$$

$$= \sup_{z\in\mathbb{D}}\sup_{w\in\mathbb{D}} (1-|z|^2)^{\gamma}(1-|w|^2)^{2+\frac{\alpha}{2}-\gamma}|(k_z^{\alpha})'(w)|$$

$$= \sup_{z\in\mathbb{D}}\sup_{w\in\mathbb{D}} (1-|z|^2)^{\gamma}(1-|w|^2)^{2+\frac{\alpha}{2}-\gamma}\Big|(2+\alpha)\frac{(1-|z|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{z}w)^{3+\alpha}}\Big|$$

$$\leq \sup_{z\in\mathbb{D}}\sup_{w\in\mathbb{D}} (1-|w|^2)^{2+\frac{\alpha}{2}-\gamma} 2^{\gamma+1+\frac{\alpha}{2}}(1-|z|)^{\gamma}(2+\alpha)(1-|z|)^{1+\frac{\alpha}{2}}(1-|z|)^{-3-\alpha}$$

$\preceq \sup_{z\in\mathbb{D}}\sup_{w\in\mathbb{D}} (1-|w|^2)^{2+\frac{\alpha}{2}-\gamma}(1-|z|)^{\gamma-2-\frac{\alpha}{2}}$, and hence $1 \in E_{2+\frac{\alpha}{2}}$. Since $||U_z^{\alpha}f||_{E_\gamma} = \sup_{z\in\mathbb{D}} (1-|z|^2)^{\gamma}||U_z^{\alpha}(U_z^{\alpha}f)||_{2+\frac{\alpha}{2}-\gamma} = \sup_{z\in\mathbb{D}} (1-|z|^2)^{\gamma}$ $||f||_{2+\frac{\alpha}{2}-\gamma}$, $U_z^{\alpha}(B_\beta) \subset E_\gamma$ for $\gamma = 2+\frac{\alpha}{2}-\beta \geq \beta$ and hence $k_z^{\alpha} \in E_{2+\frac{\alpha}{2}-\beta}$. Moreover, $U_z^{\alpha} : B_\beta \to E_\gamma$ is an isometry whenever $\beta = \gamma$. Suppose $\gamma' < \gamma$, where $\gamma = 2+\frac{\alpha}{2}-\beta$ and $\gamma' = 2+\frac{\alpha}{2}-\beta'$. Then $\beta < \beta'$ and hence

$B_{\beta'} \subset B_\beta$. Thus $U_z^\alpha(B_{\beta'}) \subset E_\gamma$. Moreover, $T_u^\alpha(B_\beta) \subset B_{2+\beta+\frac{\alpha}{2}+\frac{\alpha+2}{s'}} \subset E_{2+\frac{\alpha}{2}-(2+\beta+\frac{\alpha}{2}-\frac{\alpha+2}{s'})} = E_{\frac{\alpha+2}{s'}-\beta}$. Thus $U_z^\alpha T_u^\alpha(B_\beta) \subset U_z^\alpha(B_{2+\frac{\alpha}{2}+\beta-\frac{\alpha+2}{s'}}) \subset E_{\frac{\alpha+2}{s'}-\beta}$ and $U_z^\alpha T_u^\alpha(B_{\beta'}) \subset E_{\frac{\alpha+2}{s'}-\beta'}$ whenever $||u||_{WR(\alpha)}$ is finite with respect to $s \in (2,\infty)$. Suppose $\sup\limits_{z\in\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} = ||u||_{WR(\alpha)} < \infty$ for some $s \in (2,\infty)$. If $f \in B_\beta$ then $T_u^\alpha f \in B_{2+\beta+\frac{\alpha}{2}-\frac{\alpha+2}{s'}}$ and $||U_z^\alpha T_u^\alpha f||_{2+\frac{\alpha}{2}-\gamma} \preceq (1-|z|^2)^{-\gamma}||T_u^\alpha f||_{2+\frac{\alpha}{2}-\gamma}$, where $\gamma = \frac{\alpha+2}{s'} - \beta$. Since $||T_u^\alpha f||_{2+\frac{\alpha}{2}-\gamma} \preceq ||u||_{WR(\alpha)}||f||_\beta$ and hence $||T_u^\alpha f||_{E_\gamma} \preceq ||u||_{WR(\alpha)}||f||_\beta$. Thus one has the following :

Theorem 3.2. *Suppose* $\sup\limits_z ||uk_z^\alpha||_{s,\alpha} < +\infty$ *for some* $s > 2$ *and* $\dfrac{\alpha+2}{s'} \geq \beta + \dfrac{1}{2}$. *Then* $T_u^\alpha : B_\beta \to E_\gamma$ *is a bounded linear operator, where* $\gamma = \frac{\alpha+2}{s'} - \beta$ *and* $||T_u^\alpha|| \preceq ||u||_{WR}$.

Theorem 3.3. *Suppose* $\sup\limits_z ||uk_z^\alpha||_{s,\alpha} < +\infty$ *for some* $s > 2$ *and* $\alpha - \beta s' > -1$. *If* $||uk_z^\alpha||_{s,\alpha} \to 0$ *as* $z \to \partial\mathbb{D}$ *then* $T_u^\alpha : B_\beta \to E_\gamma$ *is a compact linear operator, where* $\gamma = \dfrac{\alpha+2}{s'} - \beta$.

Proof. It is enough to show that if (f_n) is a bounded sequence in B_β and converges to 0 uniformly on compact subsets of $\mathbb{D}$ then $||T_u^\alpha(f_n)||_{E_\gamma} \to 0$ as $n \to \infty$. Note that for $t = 2 + \frac{\alpha}{2} - \gamma$, $1-|z|^{2t}|(T_u^\alpha f_n)'(z)| \preceq ||uk_z^\alpha||_{s,\alpha}||f_n||_\beta$. Let $M = \sup ||f_n||_\beta$. Take any $\varepsilon > 0$. Since $\lim\limits_{z\to\partial\mathbb{D}} ||uk_z^\alpha||_{s,\alpha} = 0$, there is $r > 0$ such that $0 < r < 1$ and $\sup\limits_{|z|>r} ||uk_z^\alpha||_{s,\alpha} < \dfrac{\varepsilon}{2M}$ and hence $\sup\limits_{|z|>r}(1-|z|^2)|(T_u^\alpha f_n)'(z)| \preceq \dfrac{\varepsilon}{2}$. Since $|f_n(w)| \preceq \dfrac{||f_n||_\beta}{(1-|w|^2)^\beta}$,

$$\begin{aligned}|T_u^\alpha(f_n)(z)| &= \Big|(2+\alpha)\int_{\mathbb{D}} u(w)f_n(w)\frac{1}{(1-z\overline{w})^{2+\alpha}}dA_\alpha(w)\Big| \\ &\leq \frac{(2+\alpha)||uk_z^\alpha||_{s,\alpha}}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\int_{\mathbb{D}}\frac{||f_n||_\beta^{s'}}{(1-|w|^2)^{\beta s'}}(1-|w|^2)^\alpha dA(w)\Big)^{\frac{1}{s'}} \\ &= \frac{(2+\alpha)||uk_z^\alpha||_{s,\alpha}||f_n||_\beta}{(1-|z|^2)^{1+\frac{\alpha}{2}}}\Big(\int_{\mathbb{D}}(1-|w|^2)^{\alpha-\beta s'}dA(w)\Big)^{\frac{1}{s'}}.\end{aligned}$$

Since $\int_{\mathbb{D}} (1-|w|^2)^{\alpha-\beta s'} dA(w) < \infty$ and (f_n) converges to 0 uniformly on $\{z : |z| \le r\}$, $\lim_{n\to\infty} |T_u^\alpha(f_n)(0)| = 0$. Since $|(T_u^\alpha f_n)'(z)| \preceq \frac{||uk_z^\alpha||_{s,\alpha}||f_n||_\beta}{(1-|z|^2)^t}$ and (f_n) converges to 0 uniformly on $\{z : |z| \le r\}$, $\sup_{|z|\le r} (1-|z|^2)^t$ $|(T_u^\alpha f_n)'(z)| \to 0$ as $n \to \infty$. Then we get $\lim_{n\to\infty} T_u^\alpha(f_n)_{E_\gamma} = 0$. Thus T_u^α is a compact linear operator. □

Notice that for $f \in B_\beta$, $||T_u^\alpha(f)||_{2+\frac{\alpha}{2}+\beta-\frac{2+\alpha}{s'}} \preceq ||u||_{WR(\alpha)}||f||_\beta$ and for $g \in B_{2+\frac{\alpha}{2}+\beta-\frac{2+\alpha}{s'}}$, $||g||_{E_{\frac{\alpha+2}{s'}-\beta}} \preceq ||g||_{2+\frac{\alpha}{2}+\beta-\frac{\alpha+2}{s'}}$ and hence Theorem 2.8 is an immediate consequence of Theorem 3.3.

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