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ADDITIVE FUNCTIONAL EQUATION WITH SEVERAL VARIABLES AND ITS STABILITY

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Abstract. In this paper, we prove the generalized Hyers–Ulam stability of an n-dimensional additive functional equation, and then apply stability results to Banach modules over a unital Banach algebras.

1. Introduction

The stability problem of functional equations originated from a question of S. M. Ulam [8] concerning the stability of group homomorphisms.

Let G_1 be a group and G_2 a metric group with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a number $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists near f with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

In 1941, D. H. Hyers [5] considered the case of approximately additive mappings between Banach spaces and proved the following result. Suppose that E_1 and E_2 are Banach spaces and a mapping $f : E_1 \to E_2$ satisfies the following condition: if there is a number $\epsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E_1$, then the limit $h(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E_1$ and there exists a unique additive mapping $h : E_1 \to E_2$ such that

$$\|f(x) - h(x)\| \leq \epsilon$$

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for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x \in E_1$, then the mapping h is $\mathbb{R}$-linear.

This result was generalized by T. Aoki [2] for additive mappings and by Th. M. Rassias [7] for linear mappings by establishing an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by P. Găvruta [4] by replacing the unbounded Cauchy difference by a general control function. The stability problem of various functional equations has been studied by a number of authors since then. Recently, P. Nakmahachalasint [6] considered the following n-dimensional additive functional equation

$$f\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1}), \tag{1.1}$$

where $x_0 := x_n$ and $n \geq 2$, and then investigated its Hyers–Ulam–Rassias stability.

In this paper, we establish the general solution of the following n-dimensional additive functional equation

$$f\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} f(x_i) + \sum_{i \neq j} f(x_i - x_j), \tag{1.2}$$

where $n > 1$ is fixed, and then we investigate its generalized Hyers–Ulam stability.

2. The first result of Hyers–Ulam stability

We now present the general solution of the equation (1.2) in the class of functions between two vector spaces.

LEMMA 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation (1.2) if and only if it satisfies the Cauchy additive functional equation*

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

Proof. Suppose a mapping $f : X \to Y$ satisfies the Cauchy additive functional equation. Then, it is straightforward to show that

$$f\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} f(x_i), \quad \sum_{i \neq j} f(x_i - x_j) = \sum_{i \neq j} [f(x_i) - f(x_j)] = 0.$$

Hence, f satisfies the equation (1.2).

Suppose a mapping $f : X \to Y$ satisfies the equation (1.2). Then, by setting $x_1 = \cdots = x_n := 0$ in (1.2), we can see that $f(0) = 0$. Letting $x_1 := x$ and $x_2 = \cdots = x_n := 0$ in (1.2), we have

$$0 = (n-1)[(f(x) + f(-x)],$$

which shows that f is odd. Putting $(x_1, x_2, \cdots, x_n) := (y, x, 0, \cdots, 0)$ in (1.2), we lead to

$$\begin{aligned} f(x+y) &= f(x) + f(y) + (n-2)[f(x) + f(-x) + f(y) + f(-y)] \\ &\quad + f(y-x) + f(x-y). \end{aligned}$$

By the oddness of f, we have $f(x+y) = f(x) + f(y)$, as desired. □

From now on, we denote X and Y by a normed linear space and a Banach space, respectively. For simplicity, given mappings $f : X \to Y$ and $\varphi : X^n \to [0, \infty)$, we define a difference operator Df by

$$Df(x_1, \cdots, x_n) := \sum_{i=1}^{n} f(x_i) + \sum_{i \neq j} f(x_i - x_j) - f\Big(\sum_{i=1}^{n} x_i\Big)$$

for all $x_1, \cdots, x_n \in X$, and $\Psi_i : X \to [0, \infty)$, $\Phi_i : X \to [0, \infty)$ by

$$\begin{aligned} \Psi_i(x) &:= \frac{1}{2^{i+1}}|n^2 - 4n + 2|\|f(0)\| + \frac{(n-2)}{(n-1)2^i}\varphi(2^i x, 0, \cdots, 0) \\ &\quad + \frac{1}{2^{i+1}}\varphi(2^i x, 2^i x, 0, \ldots, 0), \quad \forall i \geq 0, \\ \Phi_i(x) &:= 2^i|n^2 - 4n + 2|\|f(0)\| + \frac{(n-2)2^{i+1}}{(n-1)}\varphi(2^{-(i+1)}x, 0, \cdots, 0) \\ &\quad + 2^i\varphi(2^{-(i+1)}x, 2^{-(i+1)}x, 0, \cdots, 0), \quad \forall i \geq 0, \end{aligned}$$

for all $x \in X$.

THEOREM 2.2. *Let $\varphi : X^n \to [0, \infty)$ be a mapping which satisfies*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x_1, \cdots, 2^i x_n)}{2^i} < \infty, \ \Big(\sum_{i=0}^{\infty} 2^i \varphi(\frac{x_1}{2^i}, \cdots, \frac{x_n}{2^i}) < \infty, \ \textit{resp.}\Big)$$

for all $x_1, \cdots, x_n \in X$. If a mapping $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n) \tag{2.1}$$

for all $x_1, \cdots, x_n \in X$, then there exists a unique additive mapping $L : X \to Y$ such that L satisfies the inequality

$$\|f(x) - L(x)\| \leq \sum_{i=0}^{\infty} \Psi_i(x), \ \Big(\|f(x) - L(x)\| \leq \sum_{i=0}^{\infty} \Phi_i(x), \ \textit{resp.}\Big) \tag{2.2}$$

where the mapping L is given by

$$L(x) = \lim_{n\to\infty} 2^{-m} f(2^m x), \ \Big(L(x) = \lim_{n\to\infty} 2^{m} f(2^{-m} x), \ \textit{resp.}\Big)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping L is $\mathbb{R}$-linear.

Proof. If $n = 2$, we set $x_1 = x_2 := x$ in (2.1), then we have

$$\begin{aligned}\|2f(x) - f(2x)\| &\leq 2\|f(0)\| + \varphi(x, x)\\ &= |n^2 - 4n + 2|\|f(0)\| + 2\frac{(n-2)}{(n-1)}\varphi(x, 0) + \varphi(x, x)\end{aligned}$$

for all $x \in X$. If $n > 2$, then we set $x_1 := x$ and $x_2 = \cdots = x_n := 0$ in (2.1) and so we have

$$\|(n^2 - 2n + 1)f(0) + (n-1)(f(x) + f(-x))\| \leq \varphi(x, 0, \cdots, 0)$$

which is simplified to

$$\|(n-1)f(0) + f(x) + f(-x)\| \leq \frac{1}{n-1}\varphi(x, 0, \cdots, 0)$$

for all $x \in X$. Setting $x_1 = x_2 := x$ and $x_3 = \cdots = x_n := 0$ in (2.1), we have

$$\begin{aligned}&\|2(n-2)[(n-1)f(0) + f(x) + f(-x)]\\ &- (n^2 - 4n + 2)f(0) + 2f(x) - f(2x)\| \leq \varphi(x, x, 0, \cdots, 0)\end{aligned}$$

for all $x \in X$. Associating the last two inequalities, one has

$$\begin{aligned}\|2f(x) - f(2x)\| \leq\ & |n^2 - 4n + 2|\|f(0)\| + 2\frac{(n-2)}{(n-1)}\varphi(x, 0, \cdots, 0)\\ &+\varphi(x, x, 0, \cdots, 0)\end{aligned}$$

for all $x \in X$ and any fixed integer $n \geq 2$. Thus, one can prove

$$\begin{aligned}(2.3) \qquad &\|f(x) - 2^{-m} f(2^m x)\| \leq \sum_{i=0}^{m-1} \|2^{-i} f(2^i x) - 2^{-(i+1)} f(2^{i+1} x)\|\\ &\leq \sum_{i=0}^{m-1} \Big[\frac{1}{2^{i+1}}|n^2 - 4n + 2|\|f(0)\| + \frac{(n-2)}{(n-1)2^i}\varphi(2^i x, 0, \cdots, 0)\\ &\qquad + \frac{1}{2^{i+1}}\varphi(2^i x, 2^i x, 0, \cdots, 0)\Big]\end{aligned}$$

for all $x \in X$ and for every positive integer m. Therefore, for every positive integers m and k with $m > k$, we obtain

$$\begin{aligned}\|2^{-k}f(2^kx)-2^{-m}f(2^mx)\| &= 2^{-k}\|f(2^kx)-2^{-(m-k)}f(2^{m-k}2^kx)\| \\ &\le \sum_{i=0}^{m-k-1}2^{-k}\Psi_i(2^kx)=\sum_{i=k}^{m-1}\Psi_i(x)\end{aligned}$$

for all $x\in X$. Since $\sum_{i=0}^{\infty}\Psi_i(x)<\infty$ and $\sum_{i=k}^{m-1}\Psi_i(x)\to 0$ as $k\to\infty$, the sequence $\{2^{-m}f(2^mx)\}$ is a Cauchy in the complete normed space Y. Thus, we may define

$$L(x):=\lim_{m\to\infty}2^{-m}f(2^mx), \quad \forall x\in X.$$

Letting $m\to\infty$ in (2.3), then we get the inequality (2.2). Replace $(x_1,\cdots,x_n)$ by $(2^mx_1,\cdots,2^mx_n)$ in (2.1) and divide it by 2^m. Taking the limit in the resulting inequality, we see that

$$L\Big(\sum_{i=1}^{n}x_i\Big)=\sum_{i=1}^{n}L(x_i)+\sum_{i\neq j}L(x_i-x_j)$$

for all $x_1,\cdots,x_n\in X$. By Lemma 2.1, the mapping L is additive. Under the assumption that f is measurable or $f(tx)$ is continuous in $t\in\mathbb{R}$ for all $x\in X$, by the same reasoning as in the proof of [7], the additive mapping $L:X\to Y$ satisfies

$$L(tx)=tL(x), \quad \forall x\in X, \forall t\in\mathbb{R}.$$

That is, L is $\mathbb{R}$-linear.

Now, we finally prove the uniqueness. Let $L':X\to Y$ be another additive mapping satisfying (2.2). Then we have

$$\begin{aligned}\|L(x)-L'(x)\| &= \frac{1}{2^m}\|L(2^mx)-L'(2^mx)\| \\ &\le \frac{1}{2^m}(\|L(2^mx)-f(2^mx)\|+\|f(2^mx)-L'(2^mx)\|) \\ &\le 2\sum_{i=0}^{\infty}\frac{\Psi_i(2^mx)}{2^m}=2\sum_{i=m}^{\infty}\Psi_i(x)\end{aligned}$$

for all $x\in X$ and all $m\in\mathbb{N}$. This series converges to 0 as $m\to\infty$. So we can conclude that $L(x)=L'(x)$ for all $x\in X$. □

COROLLARY 2.3. *Let $p\neq 1$ be a positive real number and $\theta,\delta\ge 0$ be real numbers. If a mapping $f:X\to Y$ satisfies the inequality*

$$\|Df(x_1,x_2,\cdots,x_n)\|\le \delta+\theta\sum_{i=1}^{n}\|x_i\|^p$$

for all $x_1, \dots, x_n \in X$, where $\delta = 0$ when $p > 1$, then there exists a unique additive mapping $L : X \to Y$ such that L satisfies the inequality

$$\|f(x) - L(x)\| \leq |n^2 - 4n + 2|\|f(0)\| + (\frac{3n-5}{n-1})\delta + \frac{(4n-6)2^p\theta}{(n-1)|2-2^p|}\|x\|^p$$

for all $x \in X$, where $f(0) = 0$ if $p > 1$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping L is $\mathbb{R}$-linear.

Proof. Letting $\varphi(x_1, x_2, \cdots, x_n) = \delta + \theta \sum_{i=1}^{n} \|x_i\|^p$ and applying Theorem 2.2, we get the desired result, as claimed. □

Corollary 2.3 leaves the case $p = 1$ undecided. We remark that 1 is a critical value of p to which Corollary 2.3 cannot extended. In fact, we shall show that for some $\varepsilon > 0$ one can find a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|Df(x_1, x_2, \cdots, x_n)| \leq \varepsilon \sum_{i=1}^{n} |x_i| \tag{2.4}$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{R}$, however, at the same time, there is no constant δ and no additive function $T : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$|f(x) - T(x)| \leq \delta |x| \tag{2.5}$$

for all $x \in \mathbb{R}$. The following is a modified example of Z. Gajda's example [3], which illustrates that Corollary 2.3 fails to hold for $p = 1$.

Fix $\varepsilon > 0$ and put $\mu := \frac{\varepsilon}{2(n^2+1)}$. First, we define a function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) := \begin{cases} \mu & for \quad x \in [1, \infty) \\ \mu x & for \quad x \in (-1, 1) \\ -\mu & for \quad x \in (-\infty, -1]. \end{cases} \tag{2.6}$$

Evidently, ϕ is continuous and $|\phi(x)| \leq \mu$ for all $x \in \mathbb{R}$. Therefore, a function $f : \mathbb{R} \to \mathbb{R}$ may be defined by the formula

$$f(x) := \sum_{k=0}^{\infty} \frac{\phi(2^k x)}{2^k}, \quad x \in \mathbb{R}.$$

Since f is defined by means of a uniformly convergent series of continuous functions, f itself is continuous and $|f(x)| \leq \sum_{k=0}^{\infty} \frac{\mu}{2^k} = 2\mu$.We are going to show that f satisfies the inequality (2.4). If $x_1 = x_2 = \cdots = x_n = 0$, then (2.4) is trivially fulfilled. Next, assume that $0 < |x_1| + |x_2| + \cdots + |x_n| < 1$. Then there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{2^N} \leq |x_1| + |x_2| + \cdots + |x_n| < \frac{1}{2^{N-1}}.$$

Hence, $|2^{N-1}x_i| < 1$, $|2^{N-1}(x_i - x_j)| < 1$ for all $i, j = 1, 2, \cdots, n$ and $|2^{N-1}(x_1 + x_2 + \cdots + x_n)| < 1$, which implies that for each $k \in \{0, 1, 2, \cdots, N-1\}$ the numbers $2^k x_i, 2^k(x_i - x_j)$ and $2^k(x_1 + x_2 + \cdots + x_n)$ remain in the interval $(-1, 1)$. Since ϕ is linear on this interval, we infer that

$$D\phi(2^k x_1, 2^k x_2, \cdots, 2^k x_n) = 0$$

for all $k = 0, 1, \cdots, N-1$. As a result, we get

$$\begin{aligned} \frac{|Df(x_1, x_2, \cdots, x_n)|}{|x_1| + |x_2| + \cdots + |x_n|} &\leq \sum_{k=N}^{\infty} \frac{|D\phi(2^k x_1, 2^k x_2, \cdots, 2^k x_n)|}{2^k(|x_1| + |x_2| + \cdots + |x_n|)} \\ &\leq \sum_{k=0}^{\infty} \frac{(n^2+1)\mu}{2^k 2^N(|x_1| + |x_2| + \cdots + |x_n|)} \\ &\leq 2(n^2+1)\mu = \varepsilon. \end{aligned}$$

Finally, assume that $|x_1| + |x_2| + \cdots + |x_n| \geq 1$. Then merely by virtue of the boundedness of f, we have

$$\frac{|Df(x_1, x_2, \cdots, x_n)|}{|x_1| + |x_2| + \cdots + |x_n|} \leq 2(n^2+1)\mu = \varepsilon.$$

Thus we conclude that f satisfies (2.4) for all $x_1, x_2, \cdots, x_n$.

Now, contrary to what we claim, suppose that there exist a $\delta \in [0, \infty)$ and an additive function $T : \mathbb{R} \to \mathbb{R}$ such that (2.5) holds true. Then, it follows from the continuity of f that T is bounded on some neighborhood of zero. Now, by a classical result (see e.g. [1], 2.1.1., Theorem 1) there exists a real constant c such that

$$T(x) = cx, \quad \forall x \in \mathbb{R}.$$

Hence,

$$|f(x) - cx| \leq \delta|x|, \quad \forall x \in \mathbb{R},$$

which implies that

$$|\frac{f(x)}{x}| \leq \delta + |c|, \quad \forall x \in \mathbb{R}-\{0\}.$$

On the other hand, we can choose an $N' \in \mathbb{N}$ so large that $N'\mu > \delta + |c|$. Then picking out an x from the interval $(0, \frac{1}{2^{N'-1}})$, we have $2^k x \in (0,1)$ for each $k \in \{0, 1, 2, \cdots, N'-1\}$. Consequently, for this x we have

$$\frac{f(x)}{x} \geq \sum_{k=0}^{N'-1} \mu = N'\mu > \delta + |c|,$$

which yields a contradiction. Thus the function f provides a good example to the effect that Corollary 2.3 fails to hold for $p = 1$.

3. The second result of Hyers–Ulam stability

In this part, we investigate alternative generalized Hyers–Ulam stability of the equation (1.2).

THEOREM 3.1. *If a mapping $f : X \to Y$ satisfies the inequality (2.1), and if $\varphi : X^n \to [0, \infty)$ satisfies*

$$\sum_{i=0}^{\infty} \frac{\varphi(n^i x_1, \cdots, n^i x_n)}{n^i} < \infty, \ \Big(\sum_{i=0}^{\infty} n^i \varphi(\frac{x_1}{n^i}, \cdots, \frac{x_n}{n^i}) < \infty, \ resp.\Big)$$

for all $x_1, \cdots, x_n \in X$, then there exists a unique additive mapping $L : X \to Y$ such that L satisfies the inequality

$$\|f(x) - L(x)\| \leq n\|f(0)\| + \sum_{i=0}^{\infty} \frac{1}{n^{i+1}} \varphi(n^i x, n^i x, \cdots, n^i x) \tag{3.1}$$

$$\Big(\|f(x) - L(x)\| \leq \sum_{i=0}^{\infty} n^i \varphi(\frac{x}{n^{i+1}}, \frac{x}{n^{i+1}}, \cdots, \frac{x}{n^{i+1}}), \ resp.\Big)$$

for all $x \in X$. The mapping L is given by

$$L(x) = \lim_{m\to\infty} n^{-m} f(n^m x), \ \Big(L(x) = \lim_{m\to\infty} n^m f(n^{-m} x), \ resp.\Big)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping L is $\mathbb{R}$-linear.

Proof. Setting $x_1 = \cdots = x_n := x$ in (2.1), we have

$$\|nf(x) + n(n-1)f(0) - f(nx)\| \leq \varphi(x, \cdots, x)$$

which is simplified to

$$\|nf(x) - f(nx)\| \leq n(n-1)\|f(0)\| + \varphi(x, \cdots, x) \tag{3.2}$$

for all $x \in X$. Thus

$$\begin{aligned}\|f(x) - n^{-m} f(n^m x)\| &\le \sum_{i=0}^{m-1} \|\frac{f(n^i x)}{n^i} - \frac{f(n^{i+1}x)}{n^{i+1}}\| \\ &\le \sum_{i=0}^{m-1} \Big[\frac{n-1}{n^i}\|f(0)\| + \frac{1}{n^{i+1}}\varphi(n^i x, \cdots, n^i x)\Big]\end{aligned}$$

for all $x \in X$ and all $m \ge 1$. The rest of proof is similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let $p \neq 1$ be a positive real number and $\theta, \delta \ge 0$ be real numbers. If a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, \cdots, x_n)\| \le \delta + \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \cdots, x_n \in X$, where $\delta = 0$ when $p > 1$, then there exists a unique additive mapping $L : X \to Y$ such that L satisfies the inequality

$$\|f(x) - L(x)\| \le n\|f(0)\| + \frac{\delta}{n-1} + \frac{n^p}{|n - n^p|}\|x\|^p$$

for all $x \in X$, where $f(0) = 0$ if $p > 1$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping L is $\mathbb{R}$-linear.

4. Applications to Banach modules

Throughout this section, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach B-modules with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. A linear mapping $L : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is called B-linear if

$$L(\alpha x) = \alpha L(x)$$

for all $\alpha \in B$ and $x \in {}_B\mathbb{B}_1$. We denote $D_a f$ by

$$D_a f(x_1, \cdots, x_n) := \sum_{i=1}^{n} f(ax_i) + \sum_{i \neq j} f(ax_i - ax_j) - af\Big(\sum_{i=1}^{n} x_i\Big)$$

for all $a \in B(1) := \{a \in B : |a| = 1\}$ and $x_1, \cdots, x_n \in {}_B\mathbb{B}_1$.

Theorem 4.1. *Let $\varphi : {}_B\mathbb{B}_1^n \to [0, \infty)$ be a mapping which satisfies*

$$\sum_{i=0}^{\infty} \frac{\varphi(n^i x_1, \cdots, n^i x_n)}{n^i} < \infty, \ \Big(\sum_{i=0}^{\infty} n^i \varphi(\frac{x_1}{n^i}, \cdots, \frac{x_n}{n^i}) < \infty, \ \textit{resp.}\Big)$$

for all $x_1, \cdots, x_n \in {}_B\mathbb{B}_1$. *If a mapping* $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ *satisfies the inequality*

$$\|D_a f(x_1, \cdots, x_n)\| \leq \varphi(x_1, \cdots, x_n) \tag{4.1}$$

for all $a \in B(1)$ *and* $x_1, \cdots, x_n \in {}_B\mathbb{B}_1$, *and if* f *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in {}_B\mathbb{B}_1$, *then there exists a unique* B*-linear mapping* $L : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ *such that* L *satisfies the inequality*

$$\|f(x) - L(x)\| \leq n\|f(0)\| + \sum_{i=0}^{\infty} \frac{\varphi(n^i x, \cdots, n^i x)}{n^{i+1}}$$

$$\Big(\|f(x) - L(x)\| \leq \sum_{i=0}^{\infty} n^i \varphi(\frac{x}{n^{i+1}}, \cdots, \frac{x}{n^{i+1}}), \text{ resp.}\Big)$$

for all $x \in {}_B\mathbb{B}_1$.

Proof. By Theorem 3.1, it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $L : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the inequality (3.1). Under the assumption that f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, the mapping L is $\mathbb{R}$-linear. And taking $x_1 = \cdots = x_n := x$ in (4.1), then we get

$$\|nf(ax) - af(nx)\| \leq n(n-1)\|f(0)\| + \varphi(x, x, \cdots, x) \tag{4.2}$$

for all $x \in {}_B\mathbb{B}_1$. Dividing (4.2) by n^m and replacing $x := n^{m-1}x (m \in \mathbb{N})$, we get $L(ax) = aL(x)$ for all $x \in {}_B\mathbb{B}_1$ and all $a \in B(1)$ by taking $m \to \infty$. The last relation is trivially true for $a = 0$. For each element $\alpha (\neq 0) \in B$, $\alpha = |\alpha| \cdot \frac{\alpha}{|\alpha|}$ and $\frac{\alpha}{|\alpha|} \in B(1)$. Since L is $\mathbb{R}$-linear, we see

$$L(\alpha x) = L\Big(|\alpha| \cdot \frac{\alpha}{|\alpha|} x\Big) = |\alpha| L\Big(\frac{\alpha}{|\alpha|} x\Big) = |\alpha| \cdot \frac{\alpha}{|\alpha|} L(x) = \alpha L(x)$$

for each nonzero $\alpha \in B$ and all $x \in {}_B\mathbb{B}_1$. So the unique $\mathbb{R}$-linear mapping L is also B-linear, as desired. □

COROLLARY 4.2. *Let* $p \neq 1$ *be a positive real number and* $\theta, \delta \geq 0$ *be real numbers. If a mapping* $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ *satisfies the inequality*

$$\|D_a f(x_1, x_2, \cdots, x_n)\| \leq \delta + \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $a \in B(1)$ *and all* $x_1, x_2, \cdots, x_n \in {}_B\mathbb{B}_1$, *where* $\delta = 0$ *when* $p > 1$, *and if* f *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in$

${}_B\mathbb{B}_1$, *then there exists a unique* B*-linear mapping* $L : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ *such that* L *satisfies the inequality*

$$\|f(x) - L(x)\| \leq n\|f(0)\| + \frac{\delta}{n-1} + \frac{n^p}{|n - n^p|}\|x\|^p$$

for all $x \in {}_B\mathbb{B}_1$.

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