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GENERALIZED HYERS—ULAM—RASSIAS STABILITY OF A FUNCTIONAL EQUATION IN THREE VARIABLES

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Abstract. In this paper, we prove the generalized Hyers–Ulam–Rassias stability of the functional equation

$$af\left(\frac{x+y+z}{b}\right)+af\left(\frac{x-y+z}{b}\right)+af\left(\frac{x+y-z}{b}\right)$$
$$+af\left(\frac{-x+y+z}{b}\right)=cf(x)+cf(y)+cf(z).$$

1. Introduction

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [4] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x)-T(x)\| \le \epsilon$$

for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(||x||^p + ||y||^p)$$

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for all $x, y \in X$. Th.M. Rassias [7] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in X$. Găvruta [3] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$. The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [8] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [2], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation.

In [5], the authors solved the quadratic type functional equation

$$\begin{aligned} a^2 f(\frac{x + y + z}{a}) + a^2 f(\frac{x - y + z}{a}) + a^2 f(\frac{x + y - z}{a}) \\ + a^2 f(\frac{-x + y + z}{a}) = 4f(x) + 4f(y) + 4f(z), \end{aligned}$$

and proved the Hyers–Ulam–Rassias stability of the quadratic type functional equation.

Throughout this paper, assume that a, b, c are positive real numbers, and that X and Y are a real normed vector space with norm $|| \cdot ||$ and a real Banach space with norm $\| \cdot \|$, respectively.

In [6], the authors solved the following functional equation

$$\begin{aligned} &af(\frac{x+y+z}{b}) + af(\frac{x-y+z}{b}) + af(\frac{x+y-z}{b}) \\ &\qquad + af(\frac{-x+y+z}{b}) = cf(x) + cf(y) + cf(z) \end{aligned} \tag{1.i}$$

for all $x, y, z \in X$, and prove the Hyers–Ulam–Rassias stability of the functional equation.

In this paper, we prove the generalized Hyers–Ulam–Rassias stability of the functional equation (1.i).

2. Stability of a functional equation in three variables

Given a mapping $f : X \to Y$, we set

$$\begin{aligned} Df(x,y,z) :=& af(\frac{x+y+z}{b}) + af(\frac{x-y+z}{b}) + af(\frac{x+y-z}{b}) \\ &+ af(\frac{-x+y+z}{b}) - cf(x) - cf(y) - cf(z) \end{aligned}$$

for all $x, y, z \in X$.

THEOREM 1. *Let $f : X \to Y$ be an odd mapping for which there is a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x,y,z) : = \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty, \tag{2.i}$$

$$\|Df(x,y,z)\| \le \varphi(x,y,z) \tag{2.ii}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2c}(\widetilde{\varphi}(2x,0,0) + \widetilde{\varphi}(x,x,0)) \tag{2.iii}$$

for all $x \in X$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$ since f is an odd mapping. Putting $y = z = 0$ in (2.ii) and then replacing

x by $2x$, we have

$$\|af(\frac{2x}{b}) - \frac{c}{2}f(2x)\| \le \frac{1}{2}\varphi(2x, 0, 0) \tag{2.1}$$

for all $x \in X$. Putting $y = x$ and $z = 0$ in (2.ii), we have

$$\|af(\frac{2x}{b}) - cf(x)\| \le \frac{1}{2}\varphi(x, x, 0) \tag{2.2}$$

for all $x \in X$. By (2.1) and (2.2), we have

$$\|f(2x) - 2f(x)\| \le \frac{1}{c}(\varphi(2x, 0, 0) + \varphi(x, x, 0)) \tag{2.3}$$

for all $x \in X$. By (2.3), we have

$$\|f(x) - \frac{f(2x)}{2}\| \le \frac{1}{2c}(\varphi(2x, 0, 0) + \varphi(x, x, 0)) \tag{2.4}$$

for all $x \in X$. Using (2.4), we have

$$\begin{aligned}\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\| &= \frac{1}{2^n}\|f(2^n x) - \frac{f(2 \cdot 2^n x)}{2}\| \\ &\le \frac{1}{2^{n+1}\ c}(\varphi(2^{n+1}x, 0, 0) + \varphi(2^n x, 2^n x, 0))\end{aligned} \tag{2.5}$$

for all $x \in X$ and all positive integers n. By (2.5), we have

$$\begin{aligned}\|\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}\| \le \sum_{k=m}^{n-1} \frac{1}{2^{k+1}\ c}\varphi(2^{k+1}x, 0, 0) \\ + \sum_{k=m}^{n-1} \frac{1}{2^{k+1}\ c}\varphi(2^k x, 2^k x, 0)\end{aligned} \tag{2.6}$$

for all $x \in X$ and all positive integers m and n with $m < n$. This shows that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{f(2^n x)}{2^n}\}$ converges for all $x \in X$. So we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in X$. Since $f(-x) = -f(x)$ for all $x \in X$, we have $A(-x) = -A(x)$ for all $x \in X$. Also, we get

$$\begin{aligned}\|DA(x,y,z)\| &= \lim_{n\to\infty} \frac{1}{2^n}\|Df(2^n x, 2^n y, 2^n z)\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0\end{aligned}$$

for all $x, y, z \in X$. By [6, Lemma 2], A is additive. Putting $m = 0$ and letting $n \to \infty$ in (2.6), we get (2.iii).

Now, let $A' : X \to Y$ be another additive mapping satisfying (2.iii). Then we have

$$\begin{aligned}\|A(x) - A'(x)\| &= \frac{1}{2^n}\|A(2^n x) - A'(2^n x)\| \\ &\le \frac{1}{2^n}(\|A(2^n x) - f(2^n x)\| + \|A'(2^n x) - f(2^n x)\|) \\ &\le \frac{2}{2^{n+1}c}(\widetilde{\varphi}(2^{n+1}x, 0, 0) + \widetilde{\varphi}(2^n x, 2^n x, 0)),\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A. □

THEOREM 2. *Let $f : X \to Y$ be an even mapping with $f(0) = 0$ for which there is a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\text{(2.iv)} \qquad \widetilde{\varphi_2}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y, 2^j z) < \infty,$$

$$\text{(2.v)} \qquad \|Df(x,y,z)\| \le \varphi(x,y,z)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\text{(2.vi)} \qquad \|f(x) - Q(x)\| \le \frac{1}{4c}(2\widetilde{\varphi_2}(x,x,0) + \widetilde{\varphi_2}(2x,0,0))$$

for all $x \in X$.

Proof. Putting $y = x$ and $z = 0$ in (2.v), we have

$$\|af(\frac{2x}{b}) - cf(x)\| \leq \frac{1}{2}\varphi(x, x, 0) \tag{2.7}$$

for all $x \in X$. Putting $y = z = 0$ in (2.v) and then replacing x by $2x$, we have

$$\|af(\frac{2x}{b}) - \frac{c}{4}f(2x)\| \leq \frac{1}{4}\varphi(2x, 0, 0) \tag{2.8}$$

for all $x \in X$. By (2.7) and (2.8), we have

$$\|f(2x) - 4f(x)\| \leq \frac{1}{c}(2\varphi(x, x, 0) + \varphi(2x, 0, 0)) \tag{2.9}$$

for all $x \in X$. By (2.9), we have

$$\|f(x) - \frac{f(2x)}{4}\| \leq \frac{1}{4c}(2\varphi(x, x, 0) + \varphi(2x, 0, 0)) \tag{2.10}$$

for all $x \in X$. Using (2.10), we have

$$\begin{aligned} \|\frac{f(2^n x)}{4^n} - \frac{f(2^{n+1}x)}{4^{n+1}}\| &= \frac{1}{4^n}\|f(2^n x) - \frac{f(2 \cdot 2^n x)}{4}\| \\ &\leq \frac{1}{4^{n+1}c}(2\varphi(2^n x, 2^n x, 0) + \varphi(2^{n+1}x, 0, 0)) \end{aligned} \tag{2.11}$$

for all $x \in X$ and all positive integers n. By (2.11), we have

$$\begin{aligned} \|\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}\| &\leq \sum_{k=m}^{n-1} \frac{2}{4^{k+1}c}\varphi(2^k x, 2^k x, 0) \\ &+ \sum_{k=m}^{n-1} \frac{1}{4^{k+1}c}\varphi(2^{k+1}x, 0, 0) \end{aligned} \tag{2.12}$$

for all $x \in X$ and all nonnegative integers m and n with $m < n$. This shows that the sequence $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$.

Since Y is complete, the sequence $\{\frac{f(2^n x)}{4^n}\}$ converges for all $x \in X$. So we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$. We have $Q(0) = 0$, $Q(-x) = Q(x)$ and

$$\begin{aligned}\|DQ(x,y,z)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|Df(2^n x, 2^n y, 2^n z)\| \\ &\le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z) = 0\end{aligned}$$

for all $x, y, z \in X$. By [6, Lemma 1], Q is quadratic. Putting $m = 0$ and letting $n \to \infty$ in (2.12), we get (2.vi). The proof of the uniqueness of Q is similar to the proof of Theorem 1. □

THEOREM 3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^3 \to [0, \infty)$ satisfying* (2.i) *and* (2.ii). *Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\begin{aligned}\left\|\frac{f(x)-f(-x)}{2} - A(x)\right\| \le& \frac{1}{4c}(\widetilde{\varphi}(2x,0,0) + \widetilde{\varphi}(x,x,0) \\ &+ \widetilde{\varphi}(-2x,0,0) + \widetilde{\varphi}(-x,-x,0),\end{aligned} \tag{2.vii}$$

$$\begin{aligned}\left\|\frac{f(x)+f(-x)}{2} - Q(x)\right\| \le& \frac{1}{8c}(2\widetilde{\varphi_2}(x,x,0) + \widetilde{\varphi_2}(2x,0,0) \\ &+ 2\widetilde{\varphi_2}(-x,-x,0) + \widetilde{\varphi_2}(-2x,0,0),\end{aligned} \tag{2.viii}$$

$$\begin{aligned}\|f(x) - Q(x) - A(x)\| \le& \frac{1}{4c}(\widetilde{\varphi}(2x,0,0) + \widetilde{\varphi}(x,x,0)) \\ &+ \frac{1}{4c}(\widetilde{\varphi}(-2x,0,0) + \widetilde{\varphi}(-x,-x,0)) \\ &+ \frac{1}{8c}(2\widetilde{\varphi_2}(x,x,0) + \widetilde{\varphi_2}(2x,0,0)) \\ &+ \frac{1}{8c}(2\widetilde{\varphi_2}(-x,-x,0) + \widetilde{\varphi_2}(-2x,0,0))\end{aligned} \tag{2.ix}$$

for all $x \in X$.

Proof. Let $g(x) := \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $g(-x) = -g(x)$ and

$$\|Dg(x, y, z)\| \leq \frac{1}{2}(\varphi(x, y, z) + \varphi(-x, -y, -z))$$

for all $x, y, z \in X$. By the same reasoning as in the proof of Theorem 1, there exists a unique additive mapping $A : X \to Y$ satisfying (2.vii).

Note that $\widetilde{\varphi_2}(x, y, z) < \infty$ since $\widetilde{\varphi_2}(x, y, z) < \widetilde{\varphi}(x, y, z)$.

Let $q(x) := \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $q(0) = 0$, $q(-x) = q(x)$ and

$$\|Dq(x, y, z)\| \leq \frac{1}{2}(\varphi(x, y, z) + \varphi(-x, -y, -z))$$

for all $x, y, z \in X$. By the same reasoning as in the proof of Theorem 2, there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.viii). Clearly, we have (2.ix) for all $x \in X$. □

Corollary 4. *Let θ and p $(0 < p < 1)$ be positive real numbers. Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that*

$$\|Df(x, y, z)\| \leq \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|\frac{f(x) - f(-x)}{2} - A(x)\| \leq \frac{\theta}{c}(\frac{2 + 2^p}{2 - 2^p})||x||^p,$$
$$\|\frac{f(x) + f(-x)}{2} - Q(x)\| \leq \frac{\theta}{c}(\frac{4 + 2^p}{4 - 2^p})||x||^p,$$
$$\|f(x) - Q(x) - A(x)\| \leq \frac{\theta}{c}(\frac{2 + 2^p}{2 - 2^p} + \frac{4 + 2^p}{4 - 2^p})||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y, z) = ||x||^p + ||y||^p + ||z||^p$ and apply Theorem 3. □

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