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ON π-V-RINGS AND INTERMEDIATE NORMALIZING EXTENSIONS

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Abstract. In this paper we study a ring over which every left module of finite length has an injective hull of finite length. We consider a ring that is a finite intermediate normalizing extension ring of such a ring. We also consider the subrings of such a ring.

Throughout this paper, all rings have identity and all modules are unital. Let R be a ring. For a left or right R-module, $E(M)$ denote the injective hull of M. For an R-module M, $Le_R(M)$ denote the length of M[1]. Recall that a ring R is a left V-ring if every simple left R-module is injective[4]. A left and right V-ring is called a V-ring. Rosenberg and Zelinsley[5] considered the rings over which every left module of finite length has an injective hull of finite length. Left V-rings form a special class in such rings. We will study such rings.

A ring R is called a left(right) π-V-ring if for every simple left(right) R-module M, the injective hull $E(M)$ is of finite length. Let n be a positive integer. A ring R is called a left(right) n-V-ring if, for every simple left(right) R-module M, the length of $E(M)$ is less than or equal to n. Michler and Villamayor[4] proved that R is a left V-ring if and only if every left R-module has the property that zero is an intersection of maximal submodules. We give a similar characterization for a left π-V-ring.

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THEOREM 1. [3] The following conditions are equivalent for a ring R:

(1) R is a left π-V-ring.

(2) Every left R-module M of finite length has an injective hull of finite length.

(3) For every left R-module M, the intersection of all submodules N with $Le_R(M/N) < \infty$ is zero.

PROOF. (1) $\Rightarrow$ (2) Assume that R is a left π-V-ring. Let M be a left R-module of finite length. Then M is Artinian and Noetherin. $U_1 \oplus U_2 \oplus \cdots \oplus U_k \leq_e M$ where U_i is uniform. It suffices $E(U_i)$ is finite length. Let A be a simple submodule of U_i. Then $E(A) = E(U_i)$ is of finite length.

(2) $\Rightarrow$ (1) Let M be a simple module. Then $E(M)$ is of finite length.

(1) $\Rightarrow$ (3) Let Ω denote an irredundant set of representatives of the simple left R-modules. $C = \oplus_{T\in\Omega} E(T)$ is a cogenerator. Let M be a nonzero left R-module. Then there exists an embedding $f : M \to \prod_{\alpha\in A} C_\alpha$ for some index set A where $C_\alpha = C$. For $T \in \Omega$, let $P_{\alpha,T}$ be the projection from $\prod_{\alpha\in A} C_\alpha$ to the summand $E(T)$ of C_α. Since $E(T)$ is of finite length by hypothesis, $M/\mathrm{Ker}(P_{\alpha,T} f)$ is of finite length. Since $\bigcap_{\alpha\in A, T\in\Omega} \mathrm{Ker}(P_{\alpha,T} f) = \mathrm{Ker}(f) = 0$, condition (3) is satisfied.

(3) $\Rightarrow$ (1) Let T be a simple left R-module. By hypothesis, the intersection of submodules N of the module $E(T)$ with $Le_R(E(T)/N) < \infty$ is zero. Hence there exists a submodule U of $E(T)$ such that $E(T)/N$ is of finite length and $T \cap U = 0$. Since $E(T)$ is a essential extension of T, this implies $U = 0$. Hence $E(T)$ is of finite length. $\square$

A ring S is called a finite normalizing extension of a ring R if R is a subring of S and $S = \sum_{i=1}^{k} a_i R$ with $a_i R = R a_i$ for each i. A ring S is called an excellent extension of R if S is a free normalizing extension of R with a basis that includes 1 and S is R-projective; that is, if N is an S-submodule of M_S, the condition that N_R is a direct summand of M_R implies that N_S is a direct summand of M_S.

Let S be a finite normalizing extension of R. If T is a subring of S such that $R \subset T \subset S$, then T is called an intermediate normalizing extension of R.

A short exact sequence $0 \to A \xrightarrow{\varphi} B \to C \to 0$ is the category of right R-modules is said to be pure(exact) if $0 \to A \bigotimes_R M \to B \bigotimes_R M \to C \bigotimes_R M \to 0$ is an exact sequence (of abelian groups) for any left R-module M. If this is the case, we say that $\varphi(A)$ is a pure submodule of B (or that B is a pure extension of $\varphi(A)$).

Theorem 2. Let S be a finite normalizing extension of R and T be an intermediate normalizing extension of R such that T_T is a pure submodule of S_T. If R is a left π-V-ring, then T is a left π-V-ring.

Proof. By hypothesis, there is a finite set $\{a_1, a_2, \cdots, a_k\}$ of elements of S such that $S = \sum_{i=1}^{k} a_i R$ and $a_i R = R a_i$ for each i. It is sufficient to show that for every left T-module M, the intersection of all T-submodules N with $Le_R(M/N) < \infty$ is zero. Let M be a nonzero left T-module and let N be an R-submodule of $S \bigotimes_T M$ with $Le_R(S \bigotimes_T M)/N = m < \infty$ and $Le_R(S \bigotimes_T M/a_i^{-1}N) < \infty$ where $a_i^{-1}N = \{m \in S \bigotimes_T M \mid a_i m \in N\}$. Let $b(N) = \bigcap_{i=1}^{n} a_i^{-1}N$. $Le_R(S \bigotimes_T M/b(N)) < \infty$[3]. $b(N)$ is an S-submodule of $S \bigotimes_T M$ contained in N. $b(N)$ is a T-submodule of M[3]. $Le_R(M/b(N)) < \infty$. Since R is a left π-V-ring, the intersection of R-submodules N of M with $Le_R(M/N) < \infty$ is zero. Therefore the intersection of T-

submodule N' of M with $Le_R(M/N') < \infty$ is zero. $\square$

THEOREM 3. Let S be a finite normalizing extension of a ring R and $R \subset T \subset S$ be an intermediate normalizing extension of R such that T_T is a pure submodule of S_T. If S is a left π-V-ring, then T is a left π-V-ring.

PROOF. By theorem 1.1, it is suffices to prove that for any left T-module M, the intersection of all T-submodule N with $Le_T(M/N) < \infty$ is zero. Let M be a nonzero T-module. By hypothesis, M can be viewed as an T-submodule of $S \bigotimes_T M$. Let L be an S-submodule of $S \bigotimes_T M$ with $Le_S(S \bigotimes_T M/L) < \infty$. By [2], $Le_T(S \bigotimes_T M/L) < \infty$. Hence $Le_T(M/M \cap L) < \infty$.

Since S is a left π-V-ring, the intersection of S-submodules L of $S \bigotimes_T M$ with $Le_T(S \bigotimes_T M/L) < \infty$ is zero. Therefore the intersection of T-submodules N of M with $Le_T(M/N) < \infty$ is zero. $\square$

COROLLARY 4. Let S be a finite normalizing extension of a ring R such that R_R is a pure submodule of S_R. If S is a left π-V-ring, then R is a left π-V-ring.

A ring S is a free normalizing extension of R with a basis that includes 1; that is, there is a finite set $\{a_1, \cdots, a_n\} \subseteq S$ such that $a_1 = 1, S = Ra_1 + \cdots + Ra_n, a_iR = Ra_i$ for all $i = 1, 2, \cdots, n$ and S is free with basis $\{a_1, a_2, \cdots, a_n\}$ as both a right and left R-module.

COROLLARY 5. If S is a free normalizing extension of R, then S is π-V-ring if and only if R is a π-V-ring.

PROOF. Since S is a free normalizing extension of R, R_R is pure in S_R and ${}_RR$ is pure in ${}_RS$. By corollary 4, R is a left and right π-V-ring.

The converse follows from theorem 2. $\square$

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