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# ON AN R-E-KKM THEOREM AND ITS APPLICATIONS

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Abstract. In this paper, we first introduce an $R$-$E$-KKM map in the E-convex settings, and next we prove an $R$-$E$-KKM theorem which generalizes the KKM theorem and the best proximity theorem simultaneously. As applications, a best proximity theorem and a fixed point theorem in E-convex sets are given.

## 1. Introduction

In a recent paper [5], Raj and Somasundaram introduce an $R$-KKM map which extends the notion of KKM maps in best proximity settings, and obtain the finite intersection theorem. As applications, they prove the existence of a best proximity point which is an extended version of the Fan-Browder fixed point theorem in a normed linear space. Recently, in [3], the author introduces a generalized E-KKM map using the E-convexity, and proves the finite intersection theorem for a generalized E-KKM map and fixed point theorems as applications.

In this paper, combining those two concepts in [3, 5], we first introduce the $R$-$E$-KKM map which generalizes the classical KKM map and $R$-KKM map simultaneously in the E-convex settings. Next, we prove an $R$-$E$-KKM theorem which generalizes the classical KKM Theorem and the best proximity theorem simultaneously. As applications, a best proximity theorem and a fixed point theorem in E-convex sets are given.

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## 2. Preliminaries

We begin with some notations and definitions. Let $X$ be a nonempty subset of a Hausdorff topological vector space $Y$. We shall denote by $2^X$ the family of all subsets of $X$, and for any nonempty subset $A$ of $Y$, by $co\,A$ the convex hull of $A$ in $Y$. We shall say $A$ is *compactly closed* if for each compact subset $K$ in $X$, $A \cap K$ is closed in $X$. When a multimap $T : X \to 2^Y$ is given, we shall denote $T^{-1}(y) := \{x \in X \mid y \in T(x)\}$ for each $y \in Y$. Denote by $[0,1]^n$ the Cartesian product of $n$ unit intervals $[0,1] \times \cdots \times [0,1]$, and denote the unit simplex in $[0,1]^n$ by $\Delta_{n-1}$, and simply denote $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_{n-1}$ with $\Sigma_{i=1}^n \lambda_i = 1$. Recall that a set $X$ is said to be *E-convex* [6] with respect to a map $E : Y \to Y$ if there is a mapping $E : Y \to Y$ such that $\lambda E(x) + (1-\lambda)E(y) \in X$ for each $x, y \in X$ and $\lambda \in [0,1]$.

Let $A$ and $B$ be nonempty subsets of a normed linear space $(X, ||\cdot||)$. We define a metric $d$ on $X$ by $d(x,y) := ||x-y||$ for each $x, y \in X$; and for each $x \in A$, we denote $d(x,B) := \inf_{y \in B} d(x,y)$ and $dist(A,B) := \inf_{x \in A} d(x,B)$. Then the pair $(A,B)$ is said to be *E-proximal* if for each $x \in A$, there exists $y \in B$ such that $d(E(x),E(y)) = dist(A,B)$. Then, it is clear that $(A,A)$ is an $E$-proximal pair.

From now on, we shall assume that $(X, ||\cdot||)$ is a normed linear space equipped with a given map $E : X \to X$.

Now we first introduce the general notion of $R$-$E$-KKM maps which fit into the generalized KKM theorem for best proximity point setting as follows:

Definition 2.1. Let $(A,B)$ be a nonempty pair of a normed linear space $X$ with a map $E : X \to X$. A multimap $T : A \to 2^B$ is called a *generalized R-E-KKM map* (simply, *R-E-KKM map*) on $A$ if for any finite subset $\{x_1, \ldots, x_n\} \subseteq A$, there exists a finite subset $\{y_1, \ldots, y_n\} \subseteq B$ such that

$$||E(x_i) - E(y_i)|| = dist(A,B) \quad \text{for each } i = 1, \ldots, n; \text{ and}$$

$$co\big(\{E(y_{i_1}), \ldots, E(y_{i_k})\}\big) \subseteq \bigcup_{j=1}^{k} T(x_{i_j})$$

for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$ $(1 \le k \le n)$.

Remark 2.2. If $E$ is the identity map on $X$, then an $R$-$E$-KKM map is a generalization of $R$-KKM maps in [5], and if $A = B$, then an $R$-$E$-KKM map reduces to the generalized E-KKM map in [3] since $E(x_i) = E(y_i)$ for each $i = 1, \ldots, n$. Furthermore, if $A = B$ and $E$ is the identity map on $X$, then an $R$-$E$-KKM map reduces to the generalized KKM map in [2]. When $A = B$ and $E$ is the identity map on $X$, and if we take $x_i = y_i$ for each $i = 1, \dots, n$, then the $R$-$E$-KKM map reduces to a KKM map in [4].

Now we shall give an example that there exists an $R$-$E$-KKM map which is not an E-KKM map:

EXAMPLE 2.3. *Let $X = \mathbb{R}$, $A = [0, 2]$, and $B = [0, 2]$. Let $E : X \to X$ be a mapping on $X$ defined by*

$$E(x) := \begin{cases} x, & \text{for each} \quad 0 \le x \le 1; \\ 2 - x, & \text{for each} \quad 1 < x \le 2; \\ 0, & \text{for each} \quad x \in X \setminus A; \end{cases}$$

*and the multimap $T : A \to 2^B$ be defined by*

$$T(x) := \begin{cases} [0, 1 + x], & \text{for each} \quad 0 \le x \le 1; \\ [1, x], & \text{for each} \quad 1 < x \le 2. \end{cases}$$

*Then, for each $x \in (1, 2]$, $E(x) = 2 - x \notin T(x) = [1, x]$ so that $T$ can not be an E-KKM map on $A$. Now we show that $T$ is an R-E-KKM map on $A$. Indeed, for any finite set $\{x_1, \dots, x_n\} \subseteq A$, we shall show that there exists a finite set $\{y_1, \dots, y_n\} \subseteq B$ such that for any subset $\{y_{i_1}, \dots, y_{i_k}\} \subseteq \{y_1, \dots, y_n\}$ $(1 \le k \le n)$, we have that for each $i = 1, \dots, n$, $||E(x_i) - E(y_i)|| = dist(A, B) = 0$, and*

$$co(\{E(y_{i_1}), \dots, E(y_{i_k})\}) \subseteq \bigcup_{j=1}^{k} T(x_{i_j}).$$

*First, in case of $1 < x_i \le 2$ for each $1 \le i \le n$, if we take $y_i := 2 - x_i$ for each $1 \le i \le n$, then for any subset $\{y_{i_1}, \dots, y_{i_k}\} \subseteq \{y_1, \dots, y_n\}$ $(1 \le k \le n)$, we have that for each $i = 1, \dots, n$,*

$$||E(x_i) - E(y_i)|| = ||(2 - x_i) - y_i|| = dist(A, B) = 0,$$

*and*

$$co(\{E(y_{i_1}), \dots, E(y_{i_k})\}) \subseteq [0, 1] \subseteq \bigcup_{j=1}^{k} T(y_{i_j}) = \bigcup_{j=1}^{k} [0, 1 + (2 - x_{i_k})];$$

*so that $T$ is an R-E-KKM map on $A$. Next, in case of $0 \le x_i \le 1$ for some $1 \le i \le n$, we should take $y_i := x_i$ for such $i$; and in case of $1 < x_j \le 2$ for some $1 \le j \le n$, then we should take $y_j := 2 - x_j$ for* *such $j$. Then, for any subset $\{y_{i_1},\ldots,y_{i_k}\}\subseteq\{y_1,\ldots,y_n\}$ $(i\le k\le n)$, we have that for each $i=1,\ldots,n$,*

$$||E(x_i)-E(y_i)||=dist(A,B)=0,$$

*and*

$$co(\{E(y_{i_1}),\ldots,E(y_{i_k})\})\subseteq[0,1]\subseteq\bigcup_{j=1}^{k}T(y_{i_j});$$

*so that $T$ is an $R$-$E$-KKM map on $A$.*

## 3. An $R$-$E$-KKM theorem and its applications

Now we begin with the following

THEOREM 3.1. *Let $(A,B)$ be a nonempty pair of a normed linear space $X$ with a map $E:X\to X$, and $T:A\to 2^B$ be an $R$-$E$-KKM map on $A$. If $T(x)$ is finitely closed (i.e., for each finite dimensional subspace $L$ in $X$, $T(x)\cap L$ is closed in the Euclidean topology in $L$) for each $x\in A$. Then the family of sets $\{T(x)\mid x\in A\}$ has the finite intersection property. Furthermore, if $A$ is E-convex, then for any finite subset $\{x_1,\ldots,x_n\}\subseteq A$, there exist $\hat{x}\in A$ and $\hat{y}\in\bigcap_{i=1}^{n}T(x_i)$ such that $||\hat{x}-\hat{y}||=dist(A,B)$.*

*Proof.* For any finite subset $\{x_1,\ldots,x_n\}\subseteq A$, we first show that $\bigcap_{i=1}^{n}T(x_i)\neq\emptyset$. Since $T$ is an $R$-$E$-KKM map on $A$, there exists a finite subset $\{y_1,\ldots,y_n\}\subseteq B$ with

$$||E(x_i)-E(y_i)||=dist(A,B)\qquad\text{for each}\quad i=1,\ldots,n,$$

such that for any subset $\{y_{i_1},\ldots,y_{i_k}\}\subseteq\{y_1,\ldots,y_n\}$ $(1\le k\le n)$,

$$co(\{E(y_{i_1}),\ldots,E(y_{i_k})\})\subseteq\bigcup_{j=1}^{k}T(x_{i_j})$$

holds, and in particular, $co\big(\{E(y_1),\ldots,E(y_n)\}\big)\subseteq\bigcup_{i=1}^{n}T(x_i)$.

Now we consider the $(n-1)$-simplex $\Delta_{n-1}$ with the vertices $e_1=(1,0,\cdots,0),e_2=(0,1,0,\cdots,0),\cdots,e_n=(0,0,\cdots,1)$; and define a continuous map $f:\Delta_{n-1}\to X$ by

$$f(\Sigma_{i=1}^{n}\lambda_ie_i):=\Sigma_{i=1}^{n}\lambda_iE(y_i),\quad\text{for each }(\lambda_1,\ldots,\lambda_n)\in\Delta_{n-1}.$$

Since $f(\Delta_{n-1})=co(\{E(y_1),\ldots,E(y_n)\})$ is a finite dimensional subset of $Y$ and $T(x_i)$ is nonempty finitely closed in $Y$, each $f^{-1}(T(x_i))$ is a nonempty closed subset of $\Delta_{n-1}$. Therefore, we consider the family of nonempty $n$ closed subsets $\{G_i := f^{-1}(T(x_i)) \mid i = 1, 2, \ldots, n\}$ of $\Delta_{n-1}$, and now we will show $\bigcap_{i=1}^n G_i \neq \emptyset$. Since $T$ is an $R$-$E$-KKM map, for any indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$,

$$f(\Sigma_{j=1}^k \lambda_{i_j} e_{i_j}) = \Sigma_{j=1}^k \lambda_{i_j} E(y_{i_j}) \subseteq \bigcup_{j=1}^k T(x_{i_j})$$

so that

$$\begin{aligned}\Sigma_{j=1}^k \lambda_{i_j} e_{i_j} \in f^{-1}(\bigcup_{j=1}^k T(x_{i_j})) &= \bigcup_{j=1}^k f^{-1}(T(x_{i_j})) \\ &= \bigcup_{j=1}^k G_{i_j} \ \subseteq \ \Delta_{n-1}.\end{aligned}$$

Therefore, we can apply the KKM theorem [4] to the family of closed subsets $\{G_i \mid 1 \leq i \leq n\}$ of $\Delta_{n-1}$ so that we have $\bigcap_{i=1}^n G_i \neq \emptyset$. Hence

$$\emptyset \neq \bigcap_{i=1}^n G_i = \bigcap_{i=1}^n f^{-1}(T(x_i)) = f^{-1}\big(\bigcap_{i=1}^n T(x_i)\big)$$

so that we have $\bigcap_{i=1}^n T(x_i) \neq \emptyset$.

Next, we assume that $A$ is E-convex, then we shall show that for a given finite subset $\{x_1, \ldots, x_n\} \subseteq A$, there exist $\hat{x} \in A$ and $\hat{y} \in \bigcap_{i=1}^n T(x_i)$ such that $||\hat{x} - \hat{y}|| = dist(A, B)$. Indeed, if we let $\hat{e} := \Sigma_{i=1}^n \hat{\lambda}_i e_i \in \bigcap_{i=1}^n G_i$, then $\hat{y} := f(\hat{e}) = \Sigma_{i=1}^n \hat{\lambda}_i E(y_i) \in \bigcap_{i=1}^n T(x_i) \subseteq B$. If we take $\hat{x} := \Sigma_{i=1}^n \hat{\lambda}_i E(x_i)$, then $\hat{x} \in A$ since $A$ is E-convex. Therefore, we have

$$\begin{aligned}dist(A, B) \leq dist\big(\hat{x}, \bigcap_{i=1}^n T(x_i)\big) &\leq ||\hat{x} - \hat{y}|| \\ &= ||\Sigma_{i=1}^n \hat{\lambda}_i E(x_i) - \Sigma_{i=1}^n \hat{\lambda}_i E(y_i)|| \\ &\leq \Sigma_{i=1}^n \hat{\lambda}_i \cdot ||E(x_i) - E(y_i)|| = dist(A, B)\end{aligned}$$

which completes the proof. □

Remark 3.2.

(1) Theorem 3.1 generalizes both Theorem 3.1 in [3] and Theorem 3.1 in [5] in the following aspects:
   (a) $T$ is an $R$-$E$-KKM map which generalizes an $R$-KKM map in [5] and generalized E-KKM map in [3] simultaneously;
   (b) the pair $(A, B)$ need not be proximal as in Theorem 3.1 [5];
   (c) $E$ need not be the identity map on $X$ as in Theorem 3.1 [3].

(2) In case of $T(\hat{x}) = B$ for $\hat{x} \in X$ in the conclusion of Theorem 3.1, since $\hat{y} \in B = T(\hat{x})$, we have

$$\begin{aligned} dist(A, B) &\leq d(\hat{x}, T(\hat{x})) \leq d(\hat{x}, \hat{y}) + d(\hat{y}, T(\hat{x})) \\ &= dist(A, B) + d(\hat{y}, T(\hat{x})) = dist(A, B) \end{aligned}$$

so that we have $d(\hat{x}, T(\hat{x})) = dist(A, B)$, i.e., $\hat{x}$ is the proximity point for $T$.

(3) In Theorem 3.1, if we replace the finitely closed assumption on $T(x)$ with compactly closed assumption on $T(x)$, then we can obtain the same conclusion by slight modification of the above proof.

As a consequence of Theorem 3.1, we can obtain the following which is a generalization of the KKM theorem in E-convex settings:

THEOREM 3.3. *Let $(A, B)$ be a nonempty pair of a normed linear space $X$ with a map $E : X \to X$, $A$ an E-convex set, and $T : A \to 2^B$ be an R-E-KKM map. If $T(x)$ is compactly closed for each $x \in A$, and $T(x_o)$ is compact for some $x_o \in A$, then $\bigcap_{x \in A} T(x) \neq \emptyset$, and there exist $\hat{x} \in A$ and $\hat{y} \in B$ such that $||\hat{x} - \hat{y}|| = dist(A, B)$.*

The following best proximity theorem, which includes the Fan-Browder fixed point theorem [4] in non-compact E-convex sets in normed linear spaces, can be a basic tool in proving many variational inequalities and intersection theorems in E-convex settings:

THEOREM 3.4. *Let $(A, B)$ be a nonempty E-proximal pair of a normed linear space $X$ with a map $E : X \to X$, and let $T : A \to 2^B$ be a multimap satisfying the following:*

(1) *for each $x \in A$, $T(x)$ is a compactly open proper subset of $B$;*
(2) *for each $y \in B$, $T^{-1}(y)$ is a nonempty E-convex subset of $A$;*
(3) *there exists an $y_o \in A$ such that $B \setminus T(y_o)$ is compact.*

*Then there is a best proximity point $\hat{x} \in A$ such that*

$$dist(\hat{x}, T(\hat{x})) = dist(A, B).$$

*Proof.* By the assumption (1), each $T(x)$ is a proper subset of $B$. Consider a multimap $S : A \to 2^B$ defined by

$$S(x) := B \setminus T(x) \quad \text{for each } x \in A.$$

By the assumption (1), each $S(x)$ is nonempty compactly closed in $B$, and by the assumption (3), $S(y_o)$ is compact. Note that $B = \bigcup_{x \in A} T(x)$.

In fact, for each $y \in B$, by the assumption (2), choose $x \in T^{-1}(y)$; then $y \in T(x)$. Therefore, $B = \bigcup_{x \in A} T(x)$ so that we have

$$\bigcap_{x \in A} S(x) = \bigcap_{x \in A} \big(B \setminus T(x)\big) = B \setminus \bigcup_{x \in A} T(x) = \emptyset.$$

Therefore, by Theorem 3.3, $S$ should not be an $R$-$E$-KKM map on $A$. Therefore, there must exist a finite subset $\{x_1, \ldots, x_m\} \subseteq A$ such that there exist $\{y_1, \ldots, y_m\} \subseteq B$ with $||E(x_i) - E(y_i)|| = dist(A, B)$ $(1 \leq i \leq m)$, and

$$co(\{E(y_1), \ldots, E(y_m)\}) \nsubseteq \bigcup_{i=1}^{m} T(x_i). \tag{$*$}$$

Indeed, for given $x_i \in A$ $(1 \leq i \leq m)$, since $(A, B)$ is an E-proximal pair, there exists $y_i \in B$ such that $||E(x_i) - E(y_i)|| = dist(A, B)$ for each $1 \leq i \leq m$. Then, the set $\{y_1, \ldots, y_m\} \subseteq B$ satisfies the condition $||E(x_i) - E(y_i)|| = dist(A, B)$ $(1 \leq i \leq m)$. Since $S$ is not an $R$-$E$-KKM map on $A$, the formula $(*)$ should hold. Therefore, there exists a point $\hat{y} = \Sigma_{i=1}^{m} \lambda_i E(y_i) \in co(\{E(y_1), \ldots, E(y_m)\})$ with $(\lambda_1, \ldots, \lambda_n) \in \Delta_{n-1}$ such that

$$\hat{y} = \Sigma_{i=1}^{m} \lambda_i E(y_i) \notin \bigcup_{i=1}^{m} S(x_i) = \bigcup_{i=1}^{m} \big(B \setminus T(x_i)\big) = B \setminus \bigcap_{i=1}^{m} T(x_i)$$

so that $\hat{y} \in \bigcap_{i=1}^{m} T(x_i)$. Therefore, $x_i \in T^{-1}(\hat{y})$ for each $i = 1, \ldots, m$. Since $T^{-1}(\hat{y})$ is E-convex by the assumption (2), we have

$$E\big(T^{-1}(\hat{y})\big) \subseteq co\{E\big(T^{-1}(\hat{y})\big)\} \subseteq T^{-1}(\hat{y}).$$

If we take $\hat{x} := \Sigma_{i=1}^{m} \lambda_i E(x_i) \in T^{-1}(\hat{y}) \subseteq A$, then $\hat{y} \in T(\hat{x})$ so that we have

$$\begin{aligned} dist(A, B) &\leq dist(\hat{x}, T(\hat{x})) \leq ||\hat{x} - \hat{y}|| \\ &= ||\Sigma_{i=1}^{m} \lambda_i E(x_i) - \Sigma_{i=1}^{n} \lambda_i E(y_i)|| \\ &\leq \Sigma_{i=1}^{m} \lambda_i \cdot ||E(x_i) - E(y_i)|| = dist(A, B). \end{aligned}$$

Therefore, $dist(\hat{x}, T(\hat{x})) = dist(A, B)$ which completes the proof. □

REMARK 3.5. In Theorem 3.4, when $B$ is a compact set, then each $T(x)$ is clearly open so that the assumption (3) is automatically satisfied. In this case, Theorem 3.4 generalizes the Fan-Browder fixed point theorem in non-compact E-convex settings in normed linear spaces.

When $A = B$ in Theorem 3.4, since $(A, A)$ is clearly an E-proximal pair of a normed linear space $X$, we can obtain the following fixed point theorem

Corollary 3.6. *Let $A$ be a nonempty subset of a normed linear space $X$ equipped with a map $E : X \to X$, and let $T : A \to 2^A$ be a multimap satisfying the following:*

(1) *for each $x \in A$, $T(x)$ is an open (proper) subset of $A$;*
(2) *for each $y \in A$, $T^{-1}(y)$ is a nonempty E-convex subset of $A$;*
(3) *there exists an $y_o \in A$ such that $B \setminus T(y_o)$ is compact.*

*Then there is a fixed point $\hat{x} \in A$ for $T$, i.e., $\hat{x} \in T(\hat{x})$.*

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