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EXISTENCE OF SOLUTIONS IN THE α-NORM FOR NEUTRAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. We study the existence of mild solutions for neutral differential equations with nonlocal conditions in the α-norm.

1. Introduction

We study the existence of mild solutions of semilinear neutral differential equation with nonlocal condition

$$\begin{cases} \frac{d}{dt}[x(t) - F(t, x(h_1(t)))] \\ = -A[x(t) - F(t, x(h_1(t)))] + G(t, x(h_2(t))), \ 0 \le t \le T, \\ x(0) + g(x) = x_0 \in X, \end{cases} \tag{1.1}$$

where $-A$ generates a C_0-semigroup on a Banach space X, and the neutral integrodifferential equation with nonlocal condition

$$\begin{cases} \frac{d}{dt}[x(t) + F(t, x(h_1(t)))] + Ax(t) \\ = \int_0^t B(t-s)x(s)ds + G(t, x(h_2(t))), 0 \le t \le T, \\ x(0) + g(x) = x_0 \in X, \end{cases} \tag{1.2}$$

where $-A$ generates an analytic semigroup in X.

Neutral differential equations arise in many areas of applied mathematics. For instance, the system of rigid heat conduction with finite

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wave speeds can be modelled in the form of equations of neutral type with delay [7].

Integro-differential equations can be used to describe many natural phenomena arising from many fields such as electronics, fluid dynamics, biological models, and chemical kinetics.

The work in nonlocal problem was initiated by Byszewski [3]. It is clear that if nonlocal condition is introduced to the equation, then it will also have better effect than the classical condition $x(0) = x_0$.

In this paper we prove the existence of mild solutions of Eqs (1.1) and (1.2) by the Contraction Mapping Principle.

2. Fractional power of operator

Let X be a Banach space. We assume that $-A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$. Let $Y = (D(A), ||\cdot||_Y)$ with

$$||y||_Y = ||y|| + ||Ay||,\ y \in D(A).$$

Assume that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$||T(t)|| \leq Me^{\omega t} = N, t \geq 0.$$

Without loss of generality, we assume that $\omega > 0$. If the assumption $0 \in \rho(A)$ is not satisfied, one can substitute A for $A - \sigma I$ with σ large enough so that $0 \in \rho(A - \sigma I)$ and so we can always assume that $0 \in \rho(A)$.

From the above assumptions, it is possible to define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, $D(A^\alpha)$ is a Banach space with the norm

$$||x||_\alpha = ||A^\alpha x||,\ x \in D(A^\alpha).$$

Denote the space $(D(A^\alpha), ||\cdot||_\alpha)$ by X_α.

The basic properties of fractional power A^α and its inverse $A^{-\alpha}$ are the following:

Lemma 2.1. [8] *Let $0 < \alpha < 1$. Then*

(i) $T(t) : X \to D(A^\alpha), t > 0$.

(ii) $A^\alpha T(t)x = T(t)A^\alpha x,\ x \in D(A^\alpha), t \geq 0$.

(iii) *For every $t > 0, A^\alpha T(t)$ is bounded on X and there exists $C_\alpha > 0$ such that*

$$||A^\alpha T(t)|| \leq C_\alpha \frac{e^{\omega t}}{t^\alpha}, t > 0 \qquad (2.1)$$

and on the finite intervals,

$$||A^\alpha T(t)|| \le \frac{C_\alpha}{t^\alpha},\ 0 < t \le T, 0 \le \alpha \le 1.$$

(iv) $A^{-\alpha}$ *is a bounded linear operator on* X *with* $D(A^\alpha) = \mathrm{Im}(A^{-\alpha})$.

(v) $D(A^\beta) \hookrightarrow D(A^\alpha)$ *when* $0 < \alpha < \beta < 1$.

(vi) *There exists* $N_\alpha > 0$ *such that*

$$||[T(t) - I]A^{-\alpha}|| \le N_\alpha t^\alpha, t > 0.$$

Recall that the following formulas [4],

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\alpha} \int_0^\infty t^{-\alpha}(tI + A)^{-1} dt$$

or

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt,$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\alpha t} dt.$$

Both integrals converge in the uniform operator topology. Thus, if $T(t)$ is compact for every $t > 0$, then $A^{-\alpha}$ is compact for every $0 < \alpha < 1$. Moreover, $A^{-\beta} : X \to X_\alpha$ is also compact if $0 < \alpha < \beta < 1$.

Let $||A^{-\beta}|| \le M_0$, with a positive constant M_0. We denote by $C([0,T], X_\alpha)$ the Banach space of continuous functions from $[0,T]$ to X_α with the norm

$$||x||_C = \sup_{0 \le t \le T} ||A^\alpha x||,\ x \in C([0,T], X_\alpha).$$

3. Existence of mild solutions

We consider the following semilinear neutral differential equation with nonlocal condition

$$\begin{cases} \frac{d}{dt}[x(t) - F(t, x(h_1(t)))] \\ = -A[x(t) - F(t, x(h_1(t)))] + G(t, x(h_2(t))), 0 \le t \le T, \\ x(0) + g(x) = x_0 \in X, \end{cases} \tag{3.1}$$

We impose the following hypotheses:

(H_0) $-A$ generates an analytic semigroup $\{T(t)\}_{t \ge 0}$ in a Banach space X.

(H_1) $F : [0, T] \times X_\alpha \to X_{\alpha+\beta}$ is Lipschitz for $0 < \beta < \alpha \le 1$, i.e.,

$$||F(t_1, x_1) - F(t_2, x_2)||_{\alpha+\beta} \le L_0(|t_1 - t_2| + ||x_1 - x_2||_\alpha)$$

for any $0 \le t_1, t_2 \le T, x_1, x_2 \in X_\alpha$, for some constant $L_0 > 0$.

(H_2) $G : [0, T] \times X_\alpha \to X_\alpha$ is Lipschitz with Lipschitz constant $L_1 > 0$.

(H_3) $g : C([0, T], X_\alpha) \to X_\alpha$ is Lipschiz with Lipschitz constant $L_2 > 0$, i.e.,

$$||g(u) - g(v)||_\alpha \le L_2||u - v||_C$$

for any $u, v \in C([0, T], X_\alpha)$.

(H_4) $h_i \in C([0, T], [0, T]), i = 1, 2$.

Definition 3.1. A continuous function $x : [0, T] \to X$ is called a *mild solution* of (3.1) if it is defined by

$$\begin{aligned} x(t) &= T(t)[x_0 - g(x) - F(0, x(h_1(0)))] + F(t, x(h_1(t))) \\ &\quad + \int_0^t T(t - s)G(s, x(h_2(s)))ds. \end{aligned}$$

Theorem 3.2. *Assume that assumptions (H_0)-(H_4) hold. Then Eq. (3.1) has a unique mild solution provided that*

$$M_0L_0N + M_0L_2 + NL_0 + \frac{C_\alpha}{1-\alpha}T^{1-\alpha}L_1 < 1.$$

Proof. Define the operator $\Lambda : (C([0, T], X_\alpha) \to (C([0, T], X_\alpha)$ by

$$\begin{aligned} \Lambda x(t) &= T(t)[x_0 - g(x) - F(0, x(h_1(0)))] + F(t, x(h_1(t))) \\ &\quad + \int_0^t T(t - s)G(s, x(h_2(s)))ds, \ 0 \le t \le T. \end{aligned}$$

It is clear that $\Lambda(C([0, T], X_\alpha)) \subset C([0, T], X_\alpha)$. Let $x, y \in C([0, T], X_\alpha)$ and $t \in [0, T]$. Then

$$\begin{aligned} &||\Lambda x(t) - \Lambda y(t)||_\alpha \\ &\le ||T(t)[F(0, x(h_1(0))) - F(0, y(h_1(0)))]||_\alpha \\ &\quad + ||T(t)[g(x) - g(y)]||_\alpha \\ &\quad + ||F(t, x(h_1(t))) - F(t, y(h_1(t)))||_\alpha \\ &\quad + \Big|\Big| \int_0^t T(t - s)[G(s, x(h_2(s))) - G(s, y(h_2(s)))]ds \Big|\Big|_\alpha \end{aligned}$$

$$\leq \Big|\Big|T(t)A^{-\beta}[A^{\beta}F(0,x(h_1(0))) - A^{\beta}F(0,y(h_1(0)))]\Big|\Big|$$
$$+\Big|\Big|T(t)[g(x)-g(y)]\Big|\Big|$$
$$+\Big|\Big|A^{-\beta}[A^{\beta}F(t,x(h_1(t))) - A^{\beta}F(t,y(h_1(t)))]\Big|\Big|$$
$$+\Big|\Big|\int_0^t A^{\alpha}T(t-s)A^{-\alpha}[G(s,x(h_2(s))) - G(s,y(h_2(s)))]ds\Big|\Big|$$
$$\leq M_0NL_0 \sup_{0\leq s\leq T}||x(s)-y(s)||_{\alpha} + M_0L_2 \sup_{0\leq s\leq T}||x(s)-y(s)||_{\alpha}$$
$$+NL_0 \sup_{0\leq s\leq T}||x(s)-y(s)||_{\alpha}$$
$$+\frac{C_{\alpha}}{1-\alpha}T^{1-\alpha}L_1 \sup_{0\leq s\leq T}||x(s)-y(s)||_{\alpha}$$
$$\leq \Big(M_0NL_0 + M_0L_2 + NL_0 + \frac{C_{\alpha}}{1-\alpha}T^{1-\alpha}L_1\Big)||x-y||_C.$$

Thus Λ is a contraction on $C([0,T],X_{\alpha})$. Hence there exists a unique fixed point of Λ, which is a mild solution of Eq. (3.1). This completes the proof. □

Now, we consider the neutral integrodifferential equation with nonlocal condition

$$\begin{cases} \frac{d}{dt}[x(t)+F(t,x(h_1(t)))] + Ax(t) \\ = \int_0^t B(t-s)x(s)ds + G(t,x(h_2(t))),\ 0\leq t\leq T, \\ x(0)+g(x) = x_0 \end{cases} \tag{3.2}$$

with the following conditions: (H_0) and

(H_5) $B(t)\in L(X_{\alpha+\beta},X), 0\leq t\leq T$, where $\alpha+\beta\leq 1$ and $0<\beta<1$,
$$||B(t)||_{\alpha+\beta}\leq M_1,\ 0\leq t\leq T.$$

(H_6) $F:[0,T]\times X_{\alpha}\to X_{\alpha+\beta}$ is Lipschitz continuous with Lipschitz constant $L_3>0$ and
$$||F(t,x)||_{\alpha+\beta}\leq L_3(||x||_{\alpha}+1).$$

(H_7) $G:[0,T]\times X_{\alpha}\to X$ is Lipschitz continuous with Lipschitz constant $L_4>0$ and
$$||G(t,x)||\leq L_4(||x||_{\alpha}+1).$$

(H_8) $g: C([0,T],X_{\alpha})\to X_{\alpha}$ is continuous and satisfies Lipschitz condition with Lipschitz constant $L_5>0$, i.e.,
$$||g(u)-g(v)||_{\alpha}\leq L_5||u-v||_C \text{ for } u,v\in C([0,T],X_{\alpha}).$$

Also, g satisfies

$$||g(u)||_\alpha \leq L_5(||u||_C + 1).$$

(H_9) $h_i \in C([0,T],[0,T]), i = 1, 2.$

DEFINITION 3.3. A family of bounded linear operators $R(t) \in L(X)$ for $t \in [0,T]$ is called *resolvent operators* for

$$\begin{cases} x'(t) + Ax(t) = \int_0^t B(t-s)x(s)ds \\ x(0) = x_0 \in X \end{cases}$$

if

(i) $R(0) = I$ and $||R(t)|| \leq N_1$ for some constant $N_1 > 0$.
(ii) For all $x \in X, R(t)x$ is continuous for $t \in [0,T]$.
(iii) $R(t) \in L(D(A))$ for $t \in [0,T]$. For $x \in D(A))$,

$$R(t)x \in C^1([0,T],X) \cap C([0,T],D(A))$$

and for $t \geq 0$ such that

$$\begin{aligned} R'(t)x + AR(t) &= \int_0^t B(t-s)R(s)xds, \\ R'(t)x + R(t)Ax &= \int_0^t R(t-s)B(s)xds \end{aligned}$$

The resolvent operator, replacing role of C_0 semigroup for evolution equations, plays an important role in solving Eq. (3.2).

DEFINITION 3.4. $x \in C([0,T],X_\alpha)$ is called a *mild solution* of Eq. (3.2) if it is defined by

$$\begin{aligned} x(t) &= R(t)[x_0 + F(0,x(h_1(0))) - g(x)] - F(t,x(h_1(t))) \\ &\quad + \int_0^t R(t-s)[AF(s,x(h_1(s))) - \int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau \\ &\quad + G(s,x(h_2(s)))]ds, \ t \in [0,T]. \end{aligned}$$

The next theorem is given in [7, Theorem 3.2] without the proof. For the completeness, we give the proof.

THEOREM 3.5. *If we assume that*

$$\begin{aligned} \Big[M_0(N_1+1) &+ \frac{C_\alpha}{1-\alpha}T^{2-\alpha}M_1 + \frac{C_{1-\beta}}{\beta}T^\beta\Big]L_3 \\ &+ \frac{C_\alpha}{1-\alpha}T^{1-\alpha}L_4 + N_1L_5 < 1, \end{aligned}$$

then Eq. (3.2) has a unique mild solution.

Proof. Define the operator P on $C([0,T],X_\alpha)$ by

$$\begin{aligned} Px(t) &= R(t)[x_0 + F(0,x(h_1(0))) - g(x)] - F(t,x(h_1(t))) \\ &+ \int_0^t R(t-s)[AF(s,x(h_1(s))) \\ &- \int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau \\ &+ G(s,x(h_2(s)))]ds,\ 0 \le t \le T. \end{aligned}$$

It is not difficult to show that P maps $C([0,T],X_\alpha)$ into itself. To show that P is a contraction $C([0,T],X_\alpha)$, let $x,y \in C([0,T],X_\alpha)$ and $t \in [0,T]$. Then

$$\begin{aligned} Px(t) - Py(t) &= R(t)[x_0 + F(0,x(h_1(0))) - g(x)] - F(t,x(h_1(t))) \\ &+ \int_0^t R(t-s)[AF(s,x(h_1(s))) \\ &- \int_0^s B(s-\tau)F(\tau,x(h_1(\tau)))d\tau \\ &+ G(s,x(h_2(s)))]ds \\ &- R(t)[x_0 + F(0,y(h_1(0))) - g(y)] + F(t,y(h_1(t))) \\ &- \int_0^t R(t-s)[AF(s,y(h_1(s))) \\ &+ \int_0^s B(s-\tau)F(\tau,y(h_1(\tau)))d\tau \\ &- G(s,y(h_2(s)))]ds. \end{aligned}$$

Thus

$$\begin{aligned} &||Px(t) - Py(t)||_\alpha \\ &\le ||R(t)[F(0,x(h_1(0))) - F(0,y(h_1(0)))]||_\alpha \\ &\quad + ||R(t)[g(y) - g(x)]||_\alpha + ||F(t,y(h_1(t))) - F(t,x(h_1(t)))||_\alpha \\ &\quad + \Big|\Big| \int_0^t R(t-s)A[F(s,x(h_1(s))) - F(s,y(h_1(s)))]ds \Big|\Big|_\alpha \\ &\quad + \Big|\Big| \int_0^t R(t-s) \int_0^s B(s-\tau)[F(\tau,y(h_1(\tau))) - F(\tau,x(h_1(\tau)))]d\tau ds \Big|\Big|_\alpha \\ &\quad + \Big|\Big| \int_0^t R(t-s)[G(s,x(h_1(s))) - G(s,y(h_2(s))]ds \Big|\Big|_\alpha \\ &\le ||R(t)A^{-\beta}[A^\beta F(0,x(h_1(0))) - A^\beta F(0,y(h_1(0)))]|| \\ &\quad + ||R(t)[g(y) - g(x)]|| \end{aligned}$$

$$
\begin{aligned}
&+||A^{-\beta}||\ ||A^{\beta}F(t,y(h_1(t))) - A^{\beta}F(t,x(h_1(t)))|| \\
&+\Big|\Big|\int_0^t A^{1-\beta}R(t-s)A^{\beta}[F(s,x(h_1(s))) - F(s,y(h_1(s)))]ds\Big|\Big| \\
&+\Big|\Big|\int_0^t A^{\alpha}R(t-s)A^{-\alpha}\int_0^s B(s-\tau)[F(\tau,y(h_1(\tau))) \\
&\quad -F(\tau,x(h_1(\tau)))]d\tau ds\Big|\Big| \\
&+\Big|\Big|\int_0^t A^{\alpha}R(t-s)A^{-\alpha}[G(s,x(h_1(s)))) - G(s,y(h_2(s)))]ds\Big|\Big| \\
\le\ & N_1M_0L_3||x-y||_C + N_1L_5||x-y||_C + M_0L_3||x-y||_C \\
&+\frac{C_{1-\beta}}{\beta}T^{\beta}L_3||x-y||_C + \frac{C_\alpha}{1-\alpha}T^{2-\alpha}M_1L_3||x-y||_C \\
&+\frac{C_\alpha}{1-\alpha}T^{1-\alpha}L_4||x-y||_C \\
\le\ & \Big(\Big[M_0(N_1+1) + \frac{C_\alpha}{1-\alpha}T^{2-\alpha}M_1 + \frac{C_{1-\beta}}{\beta}T^{\beta}\Big]L_3 \\
&\quad +\frac{C_\alpha}{1-\alpha}T^{1-\alpha}L_4 + N_1L_5\Big)||x-y||_C.
\end{aligned}
$$

Hence, by the assumption, P is a contraction on $C([0,T],X_\alpha)$. Therefore the operator P has the fixed point. It implies that Eq. (3.2) has a unique mild solution. This completes the proof. □

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