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JORDAN DERIVATIONS ON PRIME RINGS AND THEIR APPLICATIONS IN BANACH ALGEBRAS, II

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Abstract. The purpose of this paper is to prove that the noncommutative version of the Singer-Wermer Conjecture is affirmative under certain conditions. Let A be a noncommutative Banach algebra. We show that if there exists a continuous linear Jordan derivation $D : A \to A$ such that $[D(x), x]D(x)^3 \in \operatorname{rad}(A)$ for all $x \in A$, then $D(A) \subseteq \operatorname{rad}(A)$.

1. Introduction

Throughout, R represents an associative ring and A will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. Let $\operatorname{rad}(R)$ denote the (*Jacobson*) *radical* of a ring R. And a ring R is said to be (*Jacobson*) *semisimple* if its Jacobson radical $\operatorname{rad}(R)$ is zero.

A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. And an additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

Johnson and Sinclair[5] have proved that any linear derivation on a semisimple Banach algebra is continuous. Singer and Wermer[13](or Theorem 16 in [1]) states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear

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derivations on a commutative semisimple Banach algebra.
Thomas[14] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

A noncommutative version of Singer and Wermer's Conjecture states that every continuous linear derivation on a noncommutative Banach algebra maps the algebra into its radical.

Vukman[16] has proved the following: Let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he has proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

Kim[6] has showed that the following result holds: Let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \to R$ such that

$$[D(x), x]D(x)[D(x), x] = 0$$

for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.

Kim[7] has showed that the following result holds: Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D(x)[D(x), x]D(x) \in \text{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \text{rad}(A)$.

For furthermore results, see the references [2, 8, 11, 15].

Kim[9] has proved the following result in the ring theory in order to apply it to the Banach algebra theory:

Let R be a 3!-torsion free semiprime ring, and suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$D(x)^2[D(x), x] = 0$$

for all $x \in R$. In this case, we obtain $[D(x), x] = 0$ for all $x \in R$. In particular, if R is a 3!-torsionfree noncommutative and prime ring, then we get $D = 0$. And using the above result, we generalize Vukman's result[16] as follows: let A be a noncommutative Banach algebra and let $D : A \longrightarrow A$ be a continuous linear Jordan derivation, and suppose that $D(x)^2[D(x), x] \in \text{rad}(A)$ holds for all $x \in A$. Then we have $D(A) \subseteq \text{rad}(A)$.

Kim[10] show that the following results hold:

Let R be a 7!-torsionfree prime ring, and if there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$D(x)^3[D(x), x] = 0$$

for all $x \in R$, then $D(x) = 0$ for all $x \in R$. Moreover, we show that if there exists a continuous linear Jordan derivation D on a a noncommutative Banach Algebra A such that

$$D(x)^3[D(x), x] \in \mathrm{rad}(A)$$

for all $x \in A$, then $D(A) \subseteq \mathrm{rad}(A)$.

In this paper, our first aim is to prove the following result in the ring theory in order to apply it to the Banach algebra theory:

Let R be a 7!-torsionfree prime ring, and suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$[D(x), x]D(x)^3 = 0$$

for all $x \in R$. In this case, we obtain $D(x) = 0$ for all $x \in R$. We apply the above result to the Banach algebra theory. Let A be a noncommutative Banach Algebra, and suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that

$$[D(x), x]D(x)^3 \in \mathrm{rad}(A)$$

for all $x \in A$. Then we obtain $D(A) \subseteq \mathrm{rad}(A)$.

2. Preliminaries

The following lemma is due to Chung and Luh[4].

Lemma 2.1. ([4] Lemma 1.) *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \cdots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \leq k \leq n$.*

The following theorem is due to Brešar[3].

Theorem 2.2. ([3] Theorem 1.) *Let R be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

3. Main results

The following lemmas are due to Kim[10].

LEMMA 3.1. ([10] Lemma 3.) *Let R be a 2-torsion free noncommutative prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[D(x), x] = 0$$

for all $x \in R$. Then we have $D(x) = 0$ for all $x \in R$.

LEMMA 3.2. ([10] Lemma 1.) *Let R be a 2-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[[D(x), x], x] = 0$$

for all $x \in R$. Then we have $[D(x), x] = 0$ for all $x \in R$.

LEMMA 3.3. ([10] Lemma 4.) *Let R be a 7!-torsionfree noncommutative prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[[D(x), x], x]yD(x)^5 = 0$$

for all $x, y \in R$. Then we have $D(x) = 0$ for all $x \in R$.

Proof. Let $[[D(x), x], x]yD(x)^5 = 0$ for all $x \in R$. Then it is obvious that $D(x)^5y[[D(x), x], x]zD(x)^5y[[D(x), x], x] = 0$ for all $x, y, z \in R$. Then since R is a 7!-torsionfree noncommutative prime ring, it follows that $D(x)^5y[[D(x), x], x] = 0$. In fact, we see that $D(x)^5y[[D(x), x], x] = 0 \iff [[D(x), x], x]yD(x)^5 = 0$ for all $x, y \in R$. Thus by Lemma 3.4 in [10], since $D(x)^5y[[D(x), x], x] = 0$ for all $x \in R$, we have $D(x) = 0$ for all $x \in R$. □

We need the following notations. After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer. when R is a ring, we shall denote the maps $B : R \times R \longrightarrow R,\ f, g : R \longrightarrow R$ by $B(x, y) \equiv [D(x), y] + [D(y), x], f(x) \equiv [D(x), x],\ g(x) \equiv [f(x), x]$ for all $x, y \in R$ respectively. And we have the basic properties:

$$B(x, y) = B(y, x),\ B(x, x) = 2f(x),\ B(x, x^2) = 2(f(x)x + xf(x)),$$
$$B(x, yz) = B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z),$$
$$B(x, xy) = 2f(x)y + xB(x, y) + D(x)[y, x],$$
$$B(x, yx) = 2yf(x) + B(x, y)x + [y, x]D(x),\ x, y, z \in R.$$

THEOREM 3.4. *Let R be a 7!-torsionfree noncommutative prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[D(x), x]D(x)^3 = 0$$

for all $x \in R$. Then we have $D(x) = 0$ for all $x \in R$.

Proof. By Theorem 2.2, we can see that D is a derivation on R. Suppose

$$f(x)D(x)^3 = 0,\ x \in R. \tag{3.1}$$

Replacing $x + ty$ for x in (3.1), we have

$$\begin{aligned}&[D(x+ty), x+ty]D(x+ty)^3 \\ &\equiv f(x)D(x)^3 + t\{B(x,y)D(x)^3 + f(x)D(y)D(x)^2 \\ &+f(x)D(x)D(y)D(x) + f(x)D(x)^2D(y)\} + t^2H_1(x,y) \\ &+t^3H_2(x,y) + t^4H_3(x,y) + t^5f(y)D(y)^3 = 0,\ x,y \in R, t \in S_3\end{aligned} \tag{3.2}$$

where $H_i, 1 \leq i \leq 3$, denotes the term satisfying the identity (3.2).
From (3.1) and (3.2), we obtain

$$\begin{aligned}&t\{B(x,y)D(x)^3 + f(x)D(y)D(x)^2 + f(x)D(x)D(y)D(x) \\ &+f(x)D(x)^2D(y)\} + t^2H_1(x,y) + t^3H_2(x,y) \\ &+t^4H_3(x,y) = 0,\ x,y \in R, t \in S_3.\end{aligned} \tag{3.3}$$

Since R is 3!-torsionfree, by Lemma 2.1 (3.3) yields

$$\begin{aligned}&B(x,y)D(x)^3 + f(x)D(y)D(x)^2 + f(x)D(x)D(y)D(x) \\ &+f(x)D(x)^2D(y) = 0, x,y \in R.\end{aligned} \tag{3.4}$$

Letting $y = x^2$ in (3.4), and using (3.1), we have

$$\begin{aligned}&2(f(x)x + xf(x))D(x)^3 + f(x)(D(x)x + xD(x))D(x)^2 \\ &+f(x)D(x)(D(x)x + xD(x))D(x) + f(x)D(x)^2(D(x)x + xD(x)) \\ &= 2g(x)D(x)^3 + 2xf(x)D(x)^3 + (g(x)D(x) + f(x)^2)D(x)^2 \\ &+g(x)D(x)^3 - f(x)D(x)^2f(x) + (g(x)D(x) + f(x)^2)D(x)^2 \\ &+f(x)D(x)^3x - f(x)D(x)^2f(x) \\ &= 2g(x)D(x)^3 + g(x)D(x)^3 + f(x)^2D(x)^2 \\ &+g(x)D(x)^3 - f(x)D(x)^2f(x) + g(x)D(x)^3 + f(x)^2D(x)^2 \\ &-f(x)D(x)^2f(x) \\ &= 5g(x)D(x)^3 + 2f(x)^2D(x)^2 - 2f(x)D(x)^2f(x) = 0, x \in R.\end{aligned} \tag{3.5}$$

Right multiplication of (3.5) by $D(x)^2$ leads to

$$\begin{aligned}&5g(x)D(x)^5 + 2f(x)^2D(x)^4 - 2f(x)D(x)^2f(x)D(x)^2 \\ &= 0, x \in R.\end{aligned} \tag{3.6}$$

Comparing (3.1) with (3.6),

$$5g(x)D(x)^5 - 2(f(x)D(x)^2)^2 = 0, x \in R. \tag{3.7}$$

On the other hand, we get from (3.1)

$$\begin{aligned} 0 &= [f(x)D(x)^3, x] \\ &= g(x)D(x)^3 + f(x)^2D(x)^2 + f(x)D(x)f(x)D(x) \\ &\quad + f(x)D(x)^2f(x), \quad x \in R. \end{aligned} \tag{3.8}$$

Right multiplication of (3.8) by $D(x)^2$ leads to

$$\begin{aligned} &g(x)D(x)^5 + f(x)^2D(x)^4 + f(x)D(x)f(x)D(x)^3 \\ &+(f(x)D(x)^2)^2 = 0, x \in R. \end{aligned} \tag{3.9}$$

Comparing (3.1), (3.7) with (3.9),

$$7(f(x)D(x)^2)^2 = 0, x \in R.$$

Since R is 7!-torsionfree, the above relation gives

$$(f(x)D(x)^2)^2 = 0, x \in R. \tag{3.10}$$

From (3.7) and (3.10),

$$5g(x)D(x)^5 = 0, x \in R.$$

Since R is 7!-torsionfree, the above relation yields

$$g(x)D(x)^5 = 0, x \in R. \tag{3.11}$$

From (3.5) and (3.8), we get

$$\begin{aligned} &4f(x)D(x)^2f(x) + 2f(x)D(x)f(x)D(x) - 3g(x)D(x)^3 \\ &= 0, x \in R. \end{aligned} \tag{3.12}$$

Writing yx for y in (3.4), we obtain

$$\begin{aligned} &f(x)D(x)^2D(y)x + f(x)D(x)^2yD(x) + f(x)D(x)D(y)xD(x) \\ &+f(x)D(x)yD(x)^2 + f(x)D(y)xD(x)^2 + f(x)yD(x)^3 \\ &+(2yf(x) + B(x,y)x + [y,x]D(x))D(x)^3 = 0, x, y \in R. \end{aligned} \tag{3.13}$$

Right multiplication of (3.4) by x leads to

$$\begin{aligned} &f(x)D(x)^2D(y)x + f(x)D(x)D(y)D(x)x + f(x)D(y)D(x)^2x \\ &+B(x,y)D(x)^3x = 0, x, y \in R. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14), we arrive at

$$\begin{aligned} &f(x)D(x)^2yD(x) - f(x)D(x)D(y)f(x) + f(x)D(x)yD(x)^2 \\ &-f(x)D(y)f(x)D(x) - f(x)D(y)D(x)f(x) + f(x)yD(x)^3 \\ &+2yf(x)D(x)^3 - B(x,y)f(x)D(x)^2 - B(x,y)D(x)f(x)D(x) \\ &-B(x,y)D(x)^2f(x) + [y,x]D(x)^4 = 0, x, y \in R. \end{aligned} \tag{3.15}$$

By (3.1) and (3.15), it is obvious that

$$\begin{aligned}(3.16)\quad & f(x)D(x)^2yD(x) - f(x)D(x)D(y)f(x) + f(x)D(x)yD(x)^2 \\ & -f(x)D(y)f(x)D(x) - f(x)D(y)D(x)f(x) + f(x)yD(x)^3 \\ & -B(x,y)f(x)D(x)^2 - B(x,y)D(x)f(x)D(x) \\ & -B(x,y)D(x)^2f(x) + [y,x]D(x)^4 = 0, x, y \in R.\end{aligned}$$

Right multiplication of (3.16) by $D(x)^3$ leads to

$$\begin{aligned}(3.17)\ & f(x)D(x)^2yD(x)^4 - f(x)D(x)D(y)f(x)D(x)^3 \\ & +f(x)D(x)yD(x)^5 - f(x)D(y)D(x)f(x)D(x)^3 \\ & -f(x)D(y)f(x)D(x)^4 + f(x)yD(x)^6 - B(x,y)D(x)^2f(x)D(x)^3 \\ & -B(x,y)D(x)f(x)D(x)^4 - B(x,y)f(x)D(x)^5 + [y,x]D(x)^7 \\ & = 0, x, y \in R.\end{aligned}$$

Combining (3.1) with (3.17), we see that

$$\begin{aligned}(3.18)\quad & f(x)D(x)^2yD(x)^4 + f(x)D(x)yD(x)^5 + f(x)yD(x)^6 \\ & +[y,x]D(x)^7 = 0, x, y \in R.\end{aligned}$$

Replacing xy for y in (3.18), it follows that

$$\begin{aligned}(3.19)\quad & f(x)D(x)^2xyD(x)^4 + f(x)D(x)xyD(x)^5 + f(x)xyD(x)^6 \\ & +x[y,x]D(x)^7 = 0, x, y \in R.\end{aligned}$$

Left multiplication of (3.18) by x leads to

$$\begin{aligned}(3.20)\quad & xf(x)D(x)^2yD(x)^4 + xf(x)D(x)yD(x)^5 + xf(x)yD(x)^6 \\ & +x[y,x]D(x)^7 = 0, x, y \in R.\end{aligned}$$

Combining (3.19) with (3.20),

$$\begin{aligned}(3.21)\quad & (g(x)D(x)^2 + f(x)^2D(x) + f(x)D(x)f(x))yD(x)^4 \\ & +(g(x)D(x) + f(x)^2)yD(x)^5 + g(x)yD(x)^6 = 0, x, y \in R.\end{aligned}$$

Writing $D(x)^4y$ for y in (3.21), we get

$$\begin{aligned}(3.22)\quad & (g(x)D(x)^6 + f(x)^2D(x)^5 + f(x)D(x)f(x)D(x)^4)yD(x)^4 \\ & +(g(x)D(x)^5 + f(x)^2D(x)^4)yD(x)^5 + g(x)D(x)^4yD(x)^6 \\ & = 0, x, y \in R.\end{aligned}$$

Left multiplication of (3.18) by $f(x)$ leads to

$$\begin{aligned}(3.23)\quad & f(x)^2D(x)^2yD(x)^4 + f(x)^2D(x)yD(x)^5 + f(x)^2yD(x)^6 \\ & +f(x)[y,x]D(x)^7 = 0, x, y \in R.\end{aligned}$$

Putting $f(x)y$ instead of y in (3.18),

$$f(x)D(x)^2f(x)yD(x)^4 + f(x)D(x)f(x)yD(x)^5 + f(x)^2yD(x)^6 + f(x)[y,x]D(x)^7 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.24}$$

Combining (3.23) with (3.24), we have

$$(f(x)D(x)^2f(x) - f(x)^2D(x)^2)yD(x)^4 + (f(x)D(x)f(x) - f(x)^2D(x))yD(x)^5 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.25}$$

Right multiplication of (3.12) by $D(x)$ leads to

$$4f(x)D(x)^2f(x)D(x) + 2f(x)D(x)f(x)D(x)^2 - 3g(x)D(x)^4 = 0, x \in R. \tag{3.26}$$

Right multiplication of (3.5) by $D(x)$ leads to

$$2f(x)D(x)^2f(x)D(x) - 2f(x)^2D(x)^3 - 5g(x)D(x)^4 = 0, x \in R. \tag{3.27}$$

From (3.1) and (3.27), we get

$$2f(x)D(x)^2f(x)D(x) - 5g(x)D(x)^4 = 0, x \in R. \tag{3.28}$$

From (3.26) and (3.28), we get

$$2f(x)D(x)f(x)D(x)^2 + 7g(x)D(x)^4 = 0, x \in R. \tag{3.29}$$

Writing $D(x)^2yg(x)$ for y in (3.21), we get

$$(g(x)D(x)^4 + f(x)^2D(x)^3 + f(x)D(x)f(x)D(x)^2)yg(x)D(x)^4 + (g(x)D(x)^3 + f(x)^2D(x)^2)yg(x)D(x)^5 + g(x)D(x)^2yg(x)D(x)^6 = 0, x, y \in R. \tag{3.30}$$

From (3.1), (3.11) and (3.30),

$$(f(x)D(x)f(x)D(x)^2 + g(x)D(x)^4)yg(x)D(x)^4 = 0, x, y \in R. \tag{3.31}$$

From (3.29), we obtain

$$(2f(x)D(x)f(x)D(x)^2 + 7g(x)D(x)^4)yg(x)D(x)^4 = 0, x, y \in R. \tag{3.32}$$

From (3.31) and (3.32),

$$5g(x)D(x)^4yg(x)D(x)^4 = 0, x, y \in R. \tag{3.33}$$

Since R is 7!-torsion-free, (3.33) gives

$$g(x)D(x)^4yg(x)D(x)^4 = 0, x, y \in R. \tag{3.34}$$

By the semiprimeness of R, (3.34) yields

$$g(x)D(x)^4 = 0, x \in R. \tag{3.35}$$

From (3.28) and (3.35), we get

$$2f(x)D(x)^2 f(x)D(x) = 0, x \in R.$$

Since R is 7!-torsion-free, the above relation gives

$$f(x)D(x)^2 f(x)D(x) = 0, x \in R. \tag{3.36}$$

From (3.29) and (3.35),

$$2f(x)D(x)f(x)D(x)^2 = 0, x \in R.$$

Since R is 7!-torsion-free, the above relation gives

$$f(x)D(x)f(x)D(x)^2 = 0, x \in R. \tag{3.37}$$

Substituting $D(x)^2 y$ for y in (3.21), it follows that

$$\begin{aligned}&(g(x)D(x)^4 + f(x)^2 D(x)^3 + f(x)D(x)f(x)D(x)^2)yD(x)^4\\&+(g(x)D(x)^3 + f(x)^2 D(x)^2)yD(x)^5 + g(x)D(x)^2 yD(x)^6\\&= 0, x, y \in R.\end{aligned} \tag{3.38}$$

From (3.1), (3.35), (3.37) and (3.38),

$$\begin{aligned}&(g(x)D(x)^3 + f(x)^2 D(x)^2)yD(x)^5 + g(x)D(x)^2 yD(x)^6\\&= 0, x, y \in R.\end{aligned} \tag{3.39}$$

Writing $D(x)y$ for y in (3.39), we get

$$\begin{aligned}&(g(x)D(x)^4 + f(x)^2 D(x)^3)yD(x)^5 + g(x)D(x)^3 yD(x)^6\\&= 0, x, y \in R.\end{aligned} \tag{3.40}$$

Combining (3.1), (3.35) with (3.40),

$$g(x)D(x)^3 yD(x)^6 = 0, x, y \in R. \tag{3.41}$$

Writing $zg(x)D(x)^3 y$ for y in (3.39), we get

$$\begin{aligned}&(g(x)D(x)^3 + f(x)^2 D(x)^2)zg(x)D(x)^3 yD(x)^5\\&+g(x)D(x)^2 zg(x)D(x)^3 yD(x)^6 = 0, x, y, z \in R.\end{aligned} \tag{3.42}$$

Combining (3.41) with (3.42),

$$\begin{aligned}&(g(x)D(x)^3 + f(x)^2 D(x)^2)zg(x)D(x)^3 yD(x)^5\\&= 0, x, y, z \in R.\end{aligned} \tag{3.43}$$

Writing $D(x)zg(x)D(x)^3y$ for y in (3.25), we get

$$(3.44)\quad \begin{aligned}&(f(x)D(x)^2f(x)D(x)-f(x)^2D(x)^3)zg(x)D(x)^3yD(x)^4\\&+(f(x)D(x)f(x)D(x)-f(x)^2D(x)^2)zg(x)D(x)^3yD(x)^5\\&+g(x)D(x)zg(x)D(x)^3yD(x)^7=0, x,y,z\in R.\end{aligned}$$

From (3.1), (3.36), (3.41) and (3.44),

$$(3.45)\quad \begin{aligned}&(f(x)D(x)f(x)D(x)-f(x)^2D(x)^2)zg(x)D(x)^3yD(x)^5\\&=0, x,y,z\in R.\end{aligned}$$

Comparing (3.43) and (3.45),

$$(3.46)\quad \begin{aligned}&(g(x)D(x)^3+f(x)D(x)f(x)D(x))zg(x)D(x)^3yD(x)^5\\&=0, x,y,z\in R.\end{aligned}$$

From (3.8) and (3.46),

$$(3.47)\quad \begin{aligned}&(g(x)D(x)^3+2f(x)^2D(x)^2\\&+f(x)D(x)^2f(x))zg(x)D(x)^3yD(x)^5=0, x,y,z\in R.\end{aligned}$$

Combining (3.5) with (3.47),

$$(3.48)\quad \begin{aligned}&(7g(x)D(x)^3+6f(x)^2D(x)^2)zg(x)D(x)^3yD(x)^5\\&=0, x,y,z\in R.\end{aligned}$$

From (3.43) and (3.48),

$$(3.49)\quad g(x)D(x)^3zg(x)D(x)^3yD(x)^5=0, x,y,z\in R.$$

Letting $yD(x)^5z$ for z in (3.49),

$$(3.50)\quad g(x)D(x)^3yD(x)^5zg(x)D(x)^3yD(x)^5=0, x,y,z\in R.$$

Hence by the semiprimeness of R, (3.50) yields

$$(3.51)\quad g(x)D(x)^3yD(x)^5=0, x,y\in R.$$

Left multiplication of (3.18) by $g(x)D(x)^3z$ leads to

$$(3.52)\quad \begin{aligned}&g(x)D(x)^3zf(x)D(x)^2yD(x)^4+g(x)D(x)^3zf(x)D(x)yD(x)^5\\&+g(x)D(x)^3zf(x)yD(x)^6+g(x)D(x)^3z[y,x]D(x)^7\\&=0, x,y,z\in R.\end{aligned}$$

From (3.51) and (3.52),

$$(3.53)\quad g(x)D(x)^3zf(x)D(x)^2yD(x)^4=0, x,y,z\in R.$$

From (3.53), we get

$$f(x)D(x)^2zg(x)D(x)^3yD(x)^4wf(x)D(x)^2zg(x)D(x)^3yD(x)^4 = 0, w, x, y, z \in R. \tag{3.54}$$

By the semiprimeness of R, (3.54) yields

$$f(x)D(x)^2zg(x)D(x)^3yD(x)^4 = 0, x, y, z \in R. \tag{3.55}$$

Replacing $f(x)z$ for z in (3.55),

$$f(x)D(x)^2f(x)zg(x)D(x)^3yD(x)^4 = 0, x, y, z \in R. \tag{3.56}$$

From (3.5) and (3.56),

$$(5g(x)D(x)^3 + 2f(x)^2D(x)^2)zg(x)D(x)^3yD(x)^4 = 0, x, y, z \in R. \tag{3.57}$$

From (3.55) and (3.57),

$$5g(x)D(x)^3zg(x)D(x)^3yD(x)^4 = 0, x, y, z \in R. \tag{3.58}$$

Replacing $5yD(x)^4z$ for z in (3.58),

$$5g(x)D(x)^3yD(x)^4z(5g(x)D(x)^3yD(x)^4) = 0, x, y, z \in R. \tag{3.59}$$

By the semiprimeness of R, (3.59) yields

$$5g(x)D(x)^3yD(x)^4 = 0, x, y \in R. \tag{3.60}$$

Since R is 7!-torsion free, (3.60) gives

$$g(x)D(x)^3yD(x)^4 = 0, x, y \in R. \tag{3.61}$$

From (3.5) and (3.61),

$$2(f(x)D(x)^2f(x) - f(x)^2D(x)^2)yD(x)^4 = 0, x, y \in R. \tag{3.62}$$

Since R is 7!-torsion free, (3.62) yields

$$(f(x)D(x)^2f(x) - f(x)^2D(x)^2)yD(x)^4 = 0, x, y \in R. \tag{3.63}$$

From (3.25) and (3.63),

$$(f(x)D(x)f(x) - f(x)^2D(x))yD(x)^5 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.64}$$

Replacing $D(x)^2y$ for y in (3.25),

$$\begin{aligned}&(f(x)D(x)^2f(x)D(x)^2 - f(x)^2D(x)^4)yD(x)^4\\&+(f(x)D(x)f(x)D(x)^2 - f(x)^2D(x)^3)yD(x)^5\\&+g(x)D(x)^2yD(x)^7 = 0, x, y \in R.\end{aligned} \tag{3.65}$$

From (3.1), and (3.36), (3.37) and (3.65),

$$g(x)D(x)^2yD(x)^7 = 0, x, y \in R. \tag{3.66}$$

Replacing $zg(x)D(x)^2y$ for y in (3.64),

$$\begin{aligned}(f(x)D(x)f(x) - f(x)^2D(x))zg(x)D(x)^2yD(x)^5 \\ +g(x)zg(x)D(x)^2yD(x)^7 = 0, x, y, z \in R.\end{aligned} \tag{3.67}$$

From (3.66) and (3.67),

$$\begin{aligned}(f(x)D(x)f(x) - f(x)^2D(x))zg(x)D(x)^2yD(x)^5 \\ = 0, x, y, z \in R.\end{aligned} \tag{3.68}$$

Replacing $D(x)z$ for z in (3.68),

$$\begin{aligned}(f(x)D(x)f(x)D(x) - f(x)^2D(x)^2)zg(x)D(x)^2yD(x)^5 \\ = 0, x, y, z \in R.\end{aligned} \tag{3.69}$$

From (3.8) and (3.69),

$$\begin{aligned}(g(x)D(x)^3 + 2f(x)^2D(x)^2 \\ +f(x)D(x)^2f(x))zg(x)D(x)^2yD(x)^5 = 0, x, y, z \in R.\end{aligned} \tag{3.70}$$

From (3.61) and (3.70),

$$\begin{aligned}(2f(x)^2D(x)^2 + f(x)D(x)^2f(x))zg(x)D(x)^2yD(x)^5 \\ = 0, x, y, z \in R.\end{aligned} \tag{3.71}$$

From (3.5) and (3.71),

$$\begin{aligned}(3f(x)D(x)^2f(x) - 5g(x)D(x)^3)zg(x)D(x)^2yD(x)^5 \\ = 0, x, y, z \in R.\end{aligned} \tag{3.72}$$

From (3.61) and (3.72),

$$3f(x)D(x)^2f(x))zg(x)D(x)^2yD(x)^5 = 0, x, y, z \in R. \tag{3.73}$$

Since R is 7!-torsion free, (3.73) yields

$$f(x)D(x)^2f(x)zg(x)D(x)^2yD(x)^5 = 0, x, y, z \in R. \tag{3.74}$$

From (3.71) and (3.74),

$$2f(x)^2D(x)^2zg(x)D(x)^2yD(x)^5 = 0, x, y, z \in R. \tag{3.75}$$

Since R is 7!-torsion free, (3.75) yields

$$f(x)^2D(x)^2zg(x)D(x)^2yD(x)^5 = 0, x, y, z \in R. \tag{3.76}$$

Replacing $zg(x)D(x)^2y$ for y in (3.21),

$$f(x)^2D(x)^2zg(x)D(x)^2yD(x)^5 + g(x)D(x)^2z \times g(x)D(x)^2yD(x)^6 = 0, x, y, z \in R. \tag{3.77}$$

From (3.76) with (3.77), we get

$$g(x)D(x)^2zg(x)D(x)^2yD(x)^6 = 0, x, y, z \in R. \tag{3.78}$$

Replacing $yD(x)^6z$ for z in (3.78),

$$g(x)D(x)^2yD(x)^6zg(x)D(x)^2yD(x)^6 = 0, x, y, z \in R. \tag{3.79}$$

By the semiprimeness of R, we obtain from (3.79),

$$g(x)D(x)^2yD(x)^6 = 0, x, y \in R. \tag{3.80}$$

From (3.1), (3.35), (3.37), (3.38), and (3.80), one obtains

$$f(x)^2D(x)^2yD(x)^5 = 0, x, y \in R. \tag{3.81}$$

Replacing $zf(x)^2D(x)^2y$ for y in (3.21),

$$(g(x)D(x)^3 + f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))zf(x)^2D(x)^2 \times yD(x)^4 + (g(x)D(x)^2 + f(x)^2D(x))zf(x)^2D(x)^2yD(x)^5 + g(x)D(x)zf(x)^2D(x)^2yD(x)^6 = 0, x, y, z \in R. \tag{3.82}$$

Combining (3.5) with (3.12),

$$7g(x)D(x)^3 + 4f(x)^2D(x)^2 + 2f(x)D(x)f(x)D(x) = 0, x \in R. \tag{3.83}$$

Combining (3.82) with (3.83),

$$(f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))z(-7g(x)D(x)^3 - 2f(x)D(x)f(x)D(x))yD(x)^4 = 0, x, y, z \in R. \tag{3.84}$$

Combining (3.61) with (3.84),

$$2(f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))zf(x)D(x)f(x)D(x)yD(x)^4 = 0, x, y, z \in R.$$

Since R is 7!-torsion-free, the above relation gives

$$(f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))zf(x)D(x)f(x)D(x)yD(x)^4 = 0, x, y, z \in R. \tag{3.85}$$

Combining (3.81) with (3.85),

$$(f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))z(f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))yD(x)^4 = 0, x, y, z \in R. \tag{3.86}$$

(3.86) yields

$$(f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))yD(x)^4z(f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))yD(x)^4 = 0, x, y, z \in R. \tag{3.87}$$

By the semiprimeness of R, we obtain from (3.87),

$$(f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))yD(x)^4 = 0, x, y \in R. \tag{3.88}$$

Combining (3.61) with (3.83),

$$(4f(x)^2D(x)^2 + 2f(x)D(x)f(x)D(x))yD(x)^4 = 0, x, y \in R. \tag{3.89}$$

Since R is 7!-torsion-free, (3.89) gives

$$(2f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))yD(x)^4 = 0, x, y \in R. \tag{3.90}$$

Combining (3.88) with (3.90),

$$f(x)^2D(x)^2yD(x)^4 = 0, x, y \in R. \tag{3.91}$$

Combining (3.88) with (3.91),

$$f(x)D(x)f(x)D(x)yD(x)^4 = 0, x, y \in R. \tag{3.92}$$

Combining (3.8), (3.61), (3.91) with (3.92), we have

$$f(x)D(x)^2f(x)yD(x)^4 = 0, x, y \in R. \tag{3.93}$$

Combining (3.25), (3.91) with (3.93),

$$(f(x)D(x)f(x) - f(x)^2D(x))yD(x)^5 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.94}$$

Replacing $D(x)y$ for y in (3.94),

$$(f(x)D(x)f(x)D(x) - f(x)^2D(x)^2)yD(x)^5 + g(x)D(x)yD(x)^7 = 0, x, y \in R. \tag{3.95}$$

Combining (3.91), (3.92) with (3.93),

$$g(x)D(x)yD(x)^7 = 0, x, y \in R. \tag{3.96}$$

Right multiplication of (3.21) by $D(x)$ leads to

$$(g(x)D(x)^2 + f(x)^2D(x) + f(x)D(x)f(x))yD(x)^5 + (g(x)D(x) + f(x)^2)yD(x)^6 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.97}$$

Combining (3.80) with (3.97),

$$(g(x)D(x)^2 + f(x)^2D(x) + f(x)D(x)f(x))yD(x)^5 + f(x)^2yD(x)^6 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.98}$$

Replacing $D(x)y$ for y in (3.98),

$$(g(x)D(x)^3 + f(x)^2D(x)^2 + f(x)D(x)f(x)D(x))yD(x)^5 + f(x)^2D(x)yD(x)^6 + g(x)D(x)yD(x)^7 = 0, x, y \in R. \tag{3.99}$$

Combining (3.61), (3.91), (3.92), (3.96) with (3.99),

$$f(x)^2D(x)yD(x)^6 = 0, x, y \in R. \tag{3.100}$$

Left multiplication of (3.18) by $f(x)^2D(x)z$ leads to

$$f(x)^2D(x)zf(x)^2D(x)yD(x)^4 + f(x)^2D(x)zf(x)D(x)yD(x)^5 + f(x)^2D(x)zf(x)yD(x)^6 + f(x)^2D(x)z[y, x]D(x)^7 = 0, x, y, z \in R. \tag{3.101}$$

Combining (3.100) with (3.101),

$$f(x)^2D(x)zf(x)^2D(x)yD(x)^4 + f(x)^2D(x)zf(x)D(x)yD(x)^5 = 0, x, y, z \in R. \tag{3.102}$$

Right multiplication of (3.102) by $D(x)$ leads to

$$f(x)^2D(x)zf(x)^2D(x)yD(x)^5 + f(x)^2D(x)zf(x)D(x)yD(x)^6 = 0, x, y, z \in R. \tag{3.103}$$

Combining (3.100) with (3.103),

$$f(x)^2D(x)zf(x)^2D(x)yD(x)^5 = 0, x, y, z \in R. \tag{3.104}$$

Replacing $yD(x)^5z$ for z in (3.104),

$$f(x)^2D(x)yD(x)^5zf(x)^2D(x)yD(x)^5 = 0, x, y, z \in R. \tag{3.105}$$

Thus by the primeness of R, (3.105) gives

$$f(x)^2D(x)yD(x)^5 = 0, \ x, y \in R. \tag{3.106}$$

Combining (3.94) with (3.106),

$$f(x)D(x)f(x)yD(x)^5 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.107}$$

Combining (3.98) with (3.106),

$$(g(x)D(x)^2 + f(x)D(x)f(x))yD(x)^5 + f(x)^2yD(x)^6 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.108}$$

Replacing $yD(x)$ for y in (3.21),

$$(g(x)D(x)^2 + f(x)^2D(x) + f(x)D(x)f(x))yD(x)^5 + (g(x)D(x) + f(x)^2)yD(x)^6 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.109}$$

Combining (3.106) with (3.109),

$$(g(x)D(x)^2 + f(x)D(x)f(x))yD(x)^5 + (g(x)D(x) + f(x)^2)yD(x)^6 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.110}$$

Combining (3.108) with (3.110),

$$g(x)D(x)yD(x)^6 = 0, x, y \in R. \tag{3.111}$$

Combining (3.111) with (3.111),

$$(g(x)D(x)^2 + f(x)D(x)f(x))yD(x)^5 + f(x)^2yD(x)^6 + g(x)yD(x)^7 = 0, x, y \in R. \tag{3.112}$$

Combining (3.107) with (3.112),

$$g(x)D(x)^2yD(x)^5 + f(x)^2yD(x)^6 = 0, x, y \in R. \tag{3.113}$$

Replacing $zg(x)D(x)^2y$ for y in (3.113),

$$g(x)D(x)^2zg(x)D(x)^2yD(x)^5 + f(x)^2zg(x)D(x)^2yD(x)^6 = 0, x, y, z \in R. \tag{3.114}$$

Combining (3.111) with (3.114),

$$g(x)D(x)^2zg(x)D(x)^2yD(x)^5 = 0, x, y, z \in R. \tag{3.115}$$

Replacing $yD(x)^5z$ for z in (3.115),

$$g(x)D(x)^2yD(x)^5zg(x)D(x)^2yD(x)^5 = 0, x, y, z \in R. \tag{3.116}$$

Thus by the primeness of R, (3.116) gives

$$g(x)D(x)^2yD(x)^5 = 0, \ x, y \in R. \tag{3.117}$$

Combining (3.113) with (3.117),

$$f(x)^2yD(x)^6 = 0, x, y \in R. \tag{3.118}$$

On the other hand, left multiplication of (3.110) by $g(x)D(x)^2z$ leads to

$$g(x)D(x)z(g(x)D(x)^2 + f(x)D(x)f(x))yD(x)^5 + g(x)D(x)^2z(g(x)D(x) + f(x)^2)yD(x)^6 + g(x)D(x)^2zg(x)yD(x)^7 = 0, x, y, z \in R. \tag{3.119}$$

From (3.111), (3.117) and (3.119), we obtain

$$g(x)D(x)^2z(g(x)D(x)^2 + f(x)D(x)f(x))yD(x)^5 = 0, x, y, z \in R. \tag{3.120}$$

From (3.118) and (3.120), we have

$$(f(x)D(x)f(x)+g(x)D(x)^2)z(f(x)D(x)f(x)+g(x)D(x)^2)yD(x)^5=0, x,y,z\in R. \tag{3.121}$$

From (3.121), we obtain

$$(f(x)D(x)f(x)+g(x)D(x)^2)yD(x)^5z(f(x)D(x)f(x)+g(x)D(x)^2)yD(x)^5=0, x,y,z\in R. \tag{3.122}$$

Since R is prime, we obtain (3.122)

$$(f(x)D(x)f(x)+g(x)D(x)^2)yD(x)^5=0, x,y\in R. \tag{3.123}$$

Replacing $z(f(x)D(x)f(x)+f(x)^2D(x)+g(x)D(x)^2)y$ for y in (3.21),

$$\begin{aligned}(f(x)D(x)f(x)&+f(x)^2D(x)+g(x)D(x)^2)z(f(x)D(x)f(x)\\&+f(x)^2D(x)+g(x)D(x)^2)yD(x)^4+(f(x)^2+g(x)D(x))z\\&\times(f(x)D(x)f(x)+f(x)^2D(x)+g(x)D(x)^2)yD(x)^5\\&+g(x)z(f(x)D(x)f(x)+f(x)^2D(x)+g(x)D(x)^2)yD(x)^6\\&=0, x,y,z\in R.\end{aligned} \tag{3.124}$$

From (3.106), (3.123) and (3.124), we get

$$(f(x)D(x)f(x)+f(x)^2D(x)+g(x)D(x)^2)z(f(x)D(x)f(x)+f(x)^2D(x)+g(x)D(x)^2)yD(x)^4=0, x,y,z\in R. \tag{3.125}$$

From (3.125), we get

$$\begin{aligned}(f(x)D(x)f(x)&+f(x)^2D(x)+g(x)D(x)^2)yD(x)^4z\\&\times(f(x)D(x)f(x)+f(x)^2D(x)+g(x)D(x)^2)yD(x)^4\\&=0, x,y,z\in R.\end{aligned} \tag{3.126}$$

Since R is prime, we obtain (3.126)

$$(f(x)D(x)f(x)+f(x)^2D(x)+g(x)D(x)^2)yD(x)^4=0, x,y\in R. \tag{3.127}$$

From (3.21) and (3.127),

$$(f(x)^2+g(x)D(x))yD(x)^5+g(x)yD(x)^6=0, x,y\in R. \tag{3.128}$$

Replacing $zg(x)D(x)y$ for y in (3.128),

$$(f(x)^2+g(x)D(x))zg(x)D(x)yD(x)^5+g(x)zg(x)D(x)yD(x)^6=0, x,y,z\in R. \tag{3.129}$$

From (3.111) and (3.129), we get

$$(f(x)^2 + g(x)D(x))zg(x)D(x)yD(x)^5 = 0, x, y, z \in R. \tag{3.130}$$

Replacing $zf(x)^2y$ for y in (3.128),

$$(f(x)^2 + g(x)D(x))zf(x)^2yD(x)^5 + g(x)zf(x)^2yD(x)^6 = 0, x, y, z \in R. \tag{3.131}$$

From (3.118) and (3.131),

$$(f(x)^2 + g(x)D(x))zf(x)^2yD(x)^5 = 0, x, y, z \in R. \tag{3.132}$$

From (3.130) and (3.132), we obtain

$$(f(x)^2 + g(x)D(x))z(f(x)^2 + D(x)g(x))yD(x)^5 = 0, x, y, z \in R. \tag{3.133}$$

From (3.133),

$$(f(x)^2 + D(x)g(x))yD(x)^5z(f(x)^2 + D(x)g(x))yD(x)^5 = 0, x, y, z \in R. \tag{3.134}$$

Since R is prime, (3.134) yields

$$(f(x)^2 + D(x)g(x))yD(x)^5 = 0, x, y \in R. \tag{3.135}$$

From (3.128) and (3.135),

$$g(x)yD(x)^6 = 0, x, y \in R. \tag{3.136}$$

Right multiplication of (3.17) by $D(x)^2$ leads to

$$\begin{aligned} &f(x)D(x)^2yD(x)^3 - f(x)D(x)D(y)f(x)D(x)^2 \\ &+f(x)D(x)yD(x)^4 - f(x)D(y)D(x)f(x)D(x)^3 \\ &-f(x)D(y)D(x)f(x)D(x)^2 + f(x)yD(x)^5 \\ &-B(x,y)f(x)D(x)^4 - B(x,y)D(x)^2f(x)D(x)^2 \\ &-B(x,y)D(x)f(x)D(x)^3 + [y,x]D(x)^6 = 0, x, y \in R. \end{aligned} \tag{3.137}$$

From (3.1) and (3.137), we get

$$\begin{aligned} &f(x)D(x)^2yD(x)^3 - f(x)D(x)D(y)f(x)D(x)^2 \\ &+f(x)D(x)yD(x)^4 - f(x)D(y)D(x)f(x)D(x)^2 + f(x)yD(x)^5 \\ &-B(x,y)D(x)^2f(x)D(x)^2 + [y,x]D(x)^6 = 0, x, y \in R. \end{aligned} \tag{3.138}$$

Left multiplication of (3.138) by $g(x)z$ leads to

$$(3.139)\quad \begin{aligned} &g(x)zf(x)D(x)^2yD(x)^3 - g(x)zf(x)D(x)D(y)f(x)D(x)^2 \\ &+g(x)zf(x)D(x)yD(x)^4 - g(x)zf(x)D(y)D(x)f(x)D(x)^2 \\ &+g(x)zf(x)yD(x)^5 - g(x)zB(x,y)D(x)^2f(x)D(x)^2 \\ &+g(x)z[y,x]D(x)^6 = 0, x, y, z \in R. \end{aligned}$$

From (3.136) and (3.139), we get

$$(3.140)\quad \begin{aligned} &g(x)zf(x)D(x)^2yD(x)^3 - g(x)zf(x)D(x)D(y)f(x)D(x)^2 \\ &+g(x)zf(x)D(x)yD(x)^4 - g(x)zf(x)D(y)D(x)f(x)D(x)^2 \\ &+g(x)zf(x)yD(x)^5 - g(x)zB(x,y)D(x)^2f(x)D(x)^2 \\ &= 0, x, y, z \in R. \end{aligned}$$

Right multiplication of (3.140) by $D(x)$ leads to

$$(3.141)\quad \begin{aligned} &g(x)zf(x)D(x)^2yD(x)^4 + g(x)zf(x)D(x)D(y)f(x)D(x)^3 \\ &+g(x)zf(x)D(x)yD(x)^5 + g(x)zf(x)D(y)D(x)f(x)D(x)^3 \\ &+g(x)zf(x)yD(x)^6 + g(x)zB(x,y)D(x)^2f(x)D(x)^3 \\ &= 0, x, y, z \in R. \end{aligned}$$

From (3.1), (3.136) and (3.141), we have

$$(3.142)\quad \begin{aligned} &g(x)zf(x)D(x)^2yD(x)^4 + g(x)zf(x)D(x)yD(x)^5 \\ &= 0, x, y, z \in R. \end{aligned}$$

Replacing $D(x)y$ for y in (3.142),

$$(3.143)\quad \begin{aligned} &g(x)zf(x)D(x)^3yD(x)^4 + g(x)zf(x)D(x)^2yD(x)^5 \\ &= 0, x, y, z \in R. \end{aligned}$$

From (3.1) and (3.143), we get

$$(3.144)\quad g(x)zf(x)D(x)^2yD(x)^5 = 0, x, y, z \in R.$$

Replacing $wf(x)D(x)^2y$ for y in (3.142),

$$(3.145)\quad \begin{aligned} &g(x)zf(x)D(x)^2wf(x)D(x)^2yD(x)^4 + g(x)zf(x)D(x)w \\ &\times f(x)D(x)^2yD(x)^5 = 0, w, x, y, z \in R. \end{aligned}$$

From (3.144) and (3.145), we have

$$(3.146)\quad g(x)zf(x)D(x)^2wf(x)D(x)^2yD(x)^4 = 0, w, x, y, z \in R.$$

From (3.146),

$$g(x)zf(x)D(x)^2yD(x)^4wg(x)zf(x)D(x)^2yD(x)^4 = 0, w, x, y, z \in R. \tag{3.147}$$

Since R is prime, we obtain from (3.147)

$$g(x)zf(x)D(x)^2yD(x)^4 = 0, x, y, z \in R. \tag{3.148}$$

From (3.142) and (3.148),

$$g(x)zf(x)D(x)yD(x)^5 = 0, x, y, z \in R. \tag{3.149}$$

Right multiplication of (3.140) by $wD(x)^5$ leads to

$$\begin{aligned} &g(x)zf(x)D(x)^2yD(x)^3wD(x)^5 + g(x)zf(x)D(x)D(y) \\ &\times f(x)D(x)^2wD(x)^5 + g(x)zf(x)D(x)yD(x)^4wD(x)^5 \\ &+g(x)zf(x)D(y)D(x)f(x)D(x)^2wD(x)^5 + g(x)zf(x)y \\ &\times D(x)^5wD(x)^5 + g(x)zB(x,y)D(x)^2f(x)D(x)^2wD(x)^5 \\ &= 0, w, x, y, z \in R. \end{aligned} \tag{3.150}$$

From (3.149) and (3.150), we have

$$g(x)zf(x)yD(x)^5wD(x)^5 = 0, w, x, y, z \in R. \tag{3.151}$$

From (3.151) and the semiprimeness of R,

$$g(x)zf(x)yD(x)^5 = 0, x, y, z \in R. \tag{3.152}$$

From (3.152) and simple calculations,

$$g(x)yD(x)^5zg(x)yD(x)^5 = 0, x, y, z \in R. \tag{3.153}$$

Since R is prime, by the semiprimeness of R, (3.153) gives

$$g(x)yD(x)^5 = 0, x, y \in R. \tag{3.154}$$

By Lemma 3.3, (3.154) gives

$$D(x) = 0, x \in R.$$

□

4. Applications in Banach algebra theory

The following theorem is proved by the same arguments as in the proof of J. Vukman's theorem [16], but it generalizes his result.

THEOREM 4.1. *Let A be a Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D(x)^3 \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. It suffices to prove the case that A is noncommutative. By the result of B.E. Johnson and A.M. Sinclair[5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair[12] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P,\ \hat{x} = x + P$. By the assumption that $[D(x), x]D(x)^3 \in \mathrm{rad}(A),\ x \in A$, we obtain $[D_P(\hat{x}), \hat{x}](D_P(\hat{x}))^3 = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 3.4 are fulfilled. Let the factor prime Banach algebra A/P be noncommutative. Then we have $D_P(\hat{x}) = 0,\ \hat{x} \in A/P$. Thus we obtain $D(x) \in P$ for all $x \in A$ and all primitive ideals of A. Hence $D(A) \subseteq \mathrm{rad}(A)$. And we consider the case that A/P is commutative. Then since A/P is a commutative Banach semisimple Banach algebra, from the result of B.E. Johnson and A.M. Sinclair[5], it follows that $D_P(\hat{x}) = 0,\ \hat{x} \in A/P$. And so, $D(x) \in P$ for all $x \in A$ and all primitive ideals of A. Hence $D(A) \subseteq \mathrm{rad}(A)$. Therefore in any case we obtain $D(A) \subseteq \mathrm{rad}(A)$. □

THEOREM 4.2. *Let A be a semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D(x)^3 = 0$$

for all $x \in A$. Then we have $D = 0$.

Proof. It suffices to prove the case that A is noncommutative. According to the result of B.E. Johnson and A.M. Sinclair[5] every linear derivation on a semisimple Banach algebra is continuous. A.M. Sinclair[12] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P,\ \hat{x} = x + P$. From the given assumptions $[D(x), x]D(x)^3 = 0,\ x \in A$, it follows that $[D_P(\hat{x}), \hat{x}](D_P(\hat{x}))^3 = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 3.4 are fulfilled. The factor algebra A/P is noncommutative, by Theorem 3.4 we have $D_P(\hat{x}) = 0,\ \hat{x} \in A/P$. Hence we get $D(A) \subseteq P$

for all primitive ideals P of A. Thus $D(A) \subseteq \mathrm{rad}(A.)$ And since A is semisimple, $D = 0$. $\square$

As a special case of Theorem 4.2 we get the following result which characterizes commutative semisimple Banach algebras.

Corollary 4.3. *Let A be a semisimple Banach algebra. Suppose*

$$[[x, y], x][x, y]^3 = 0$$

for all $x, y \in A$. In this case, A is commutative.

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Berlin-Heidelberg-New York, 1973.

[2] M. Brešar, *Derivations of noncommutative Banach algebras II*, Arch. Math. **63** (1994), 56-59.

[3] M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1003-1006.

[4] L. O. Chung and J. Luh, *Semiprime rings with nilpotent derivatives* , Canad. Math. Bull. **24** (1981), no. 4, 415-421.

[5] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 1067-1073.

[6] B. D. Kim, *On the derivations of semiprime rings and noncommutative Banach algebras*, Acta Mathematica Sinica **16** (2000), no. 1, 21-28.

[7] B. D. Kim, *Derivations of semiprime rings and noncommutative Banach algebras*, Commun. Korean Math. Soc. **17** (2002), no. 4, 607-618.

[8] B. D. Kim, *Jordan derivations of semiprime rings and noncommutative Banach algebras, I*, J. Korea Soc. Math. Educ. Ser. B. Pure Appl. Math. **15** (2008), no. 2, 179-201.

[9] B. D. Kim, *Jordan derivations of semiprime rings and noncommutative Banach algebras, II*, J. Korea Soc. Math. Educ. Ser. B. Pure Appl. Math. **15** (2008), no. 3, 259-296.

[10] B. D. Kim, *Jordan derivations of prime rings and their applications in Banach algebras, I*, Commun. Korean Math. Soc. **28** (2013), no. 3, 535-558.

[11] K. H. Park and B. D. Kim, *On continuous linear Jordan derivations of Banach algebras*, J. Korea Soc. Math. Educ. Ser. B. Pure Appl. Math. **16** (2009), no. 2, 227-241.

[12] A. M. Sinclair, *Jordan homohorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209-214.

[13] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260-264.

[14] M. P. Thomas, *The image of a derivation is contained in the radical*, Annals of Math. **128** (1988), 435-460.

[15] J. Vukman, *A result concerning derivations in noncommutative Banach algebras*, Glasnik Mathematicki **26** (1991), no. 46, 83-88.

[16] J. Vukman, *On derivations in prime rings and Banach algebras*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 877-884.

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