JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **26**, No. 1, 2013

GALOIS ACTIONS OF A CLASS INVARIANT OVER QUADRATIC NUMBER FIELDS WITH DISCRIMINANT $D \equiv 64 \pmod{72}$

Daeyeol Jeon*

Abstract. A class invariant is the value of a modular function that generates a ring class field of an imaginary quadratic number field such as the singular moduli of level 1. In this paper, we compute the Galois actions of a class invariant from a generalized Weber function $\mathfrak{g}_1$ over imaginary quadratic number fields with discriminant $D \equiv 64 \pmod{72}$.

1. Introduction

Let K be an imaginary quadratic number field with discriminant D and the ring of integers $\mathcal{O} = \mathbb{Z}[\theta]$ where

$$\theta := \begin{cases} \frac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \pmod 4; \\ \frac{-1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod 4). \end{cases}$$

Then the theory of complex multiplication states that the modular invariant $j(\mathcal{O}) = j(\theta)$ generates the *ring class field* $H_{\mathcal{O}}$ over K with degree $[H_{\mathcal{O}} : K] = h(\mathcal{O})$, the class number of $\mathcal{O}$, and the conjugates of $j(\theta)$ under the action of $Gal(H_{\mathcal{O}}/K)$ are *singular moduli* $j(\tau)$, where $\tau := \tau_Q$ is the Heegner point determined by $Q(\tau_Q, 1) = 0$ for a positive definite integral primitive binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$.

Received December 11, 2012; Accepted January 11, 2013.
2010 Mathematics Subject Classification: Primary 11F37; Secondary 11F30.
Key words and phrases: class invariants, Galois Actions, Weber functions.
This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2010-0023942).

In his Lehrbuch der Algebra [10], H. Weber calls the value of a modular function $f(\theta)$ a *class invariant* if we have

$$K(f(\theta)) = K(j(\theta)).$$

Despite a long history of the problem, one began to treat class invariants in a systemic and algorithmic way only after Shimura Reciprocity Law [8] became available. The reciprocity law provides not only a method of systematically determining whether $f(\theta)$ is a class invariant but also a description of the Galois conjugates of $f(\theta)$ under $Gal(H_{\mathcal{O}}/K)$. This tool is well illustrated in several works by Alice Gee and Peter Stevenhagen in [2, 3, 4, 9]. The author[5, 6, 7] compute the Galois actions of certain class invariants over some cases of quadratic number fields.

Gee determine the class invariants from a generalized Weber function $\mathfrak{g}_1$ by using the Shimura Reciprocity Law as follows:

THEOREM 1.1. [3, p.73, Theorem 1] *Let K be an imaginary quadratic number field with discriminant $D \equiv 64 \pmod{72}$ and the ring of integers $\mathcal{O} = \mathbb{Z}[\theta]$ where $\theta = \frac{\sqrt{D}}{2}$. Then $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$ gives an integral generator for the ring class field $H_{\mathcal{O}}$ over K where ζ_n is a primitive n-th root of unity for a positive integer n.*

In this paper, we compute the Galois actions of the class invariant $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$ under $Gal(H_{\mathcal{O}}/K)$.

2. Preliminary

Let $\mathcal{Q}_D^0$ be the set of primitive quadratic forms and $C(D) = \mathcal{Q}_D^0/\Gamma(1)$ denote the form class group of discriminant D. Since $Gal(H_{\mathcal{O}}/K)$ is isomorphic to $C(D)$, it suffices to compute the action of a primitive quadratic form $Q = [a, b, c]$ on the class invariant $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$.

THEOREM 2.1. [1, 2] *Let $\mathbb{Z}[\theta]$ be the ring of integers of an imaginary quadratic number field K with discriminant $D \equiv 0 \pmod 4$ and let $Q = [a, b, c]$ be a primitive quadratic form with discriminant D. Let $\theta = \frac{\sqrt{D}}{2}$ and $\tau_Q = \frac{-b+\sqrt{D}}{2a}$. Let $M = M_{[a,b,c]} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be given as follows:*

$$M \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \nmid a; \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \nmid c; \\ \begin{pmatrix} -\frac{b}{2}-a & -\frac{b}{2}-c \\ 1 & -1 \end{pmatrix} & \pmod{p^{r_p}} \quad \text{if } p \mid a \text{ and } p \mid c, \end{cases} \tag{2.1}$$

where p runs over all prime factors of N and $p^{r_p}||N$. Then the Galois action of the class of $[a,-b,c]$ in $C(D)$ with respect to the Artin map is given by

$$f(\theta)^{[a,-b,c]} = f^M(\tau_Q)$$

for any modular function f of level N such that $f(\theta) \in H_{\mathcal{O}}$. Here f^M denote the image of f under the action of M.

The action of M depends only on $M_{p^{r_p}}$ for all primes $p|N$ where $M_{p^{r_p}} \in \mathrm{GL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$ is the reduction modulo p^{r_p} of M. Every $M_{p^{r_p}}$ with determinant x decomposes as $M_{p^{r_p}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$ is generated by $S_{p^{r_p}} \equiv \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T_{p^{r_p}} \equiv \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, it suffices to find the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)_{p^{r_p}}$, $S_{p^{r_p}}$ and $T_{p^{r_p}}$ on f for all $p|N$, where f is a modular function of level N whose Fourier coefficients belong to $\mathbb{Q}(\zeta_N)$. For $\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)_{p^{r_p}}$, the action on f is given by lifting the automorphism of $\mathbb{Q}(\zeta_N)$ determined by

$$\zeta_{p^{r_p}} \mapsto \zeta_{p^{r_p}}^x \quad \text{and} \quad \zeta_{q^{r_q}} \mapsto \zeta_{q^{r_q}}$$

for all prime factors $q|N$ with $q \neq p$. In order that the actions of the matrices at different primes commute with each other, we lift $S_{p^{r_p}}$ and $T_{p^{r_p}}$ to matrices in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ such that they reduce to the identity matrix in $\mathrm{SL}_2(\mathbb{Z}/q^{r_q}\mathbb{Z})$ for all $q \neq p$.

The Dedekind-eta function

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \text{with} \quad q = e^{2\pi i z}$$

is holomorphic and non-zero for z in the complex upper half plane $\mathbb{H}$ and $\Delta(z) = \eta^{24}(z)$ is modular form of weight 12 with no poles or zeros on $\mathbb{H}$. Then we have generalized Weber functions as follows:

(2.2)

$$\mathfrak{g}_0(z) = \frac{\eta(\frac{z}{3})}{\eta(z)},\ \mathfrak{g}_1(z) = \zeta_{24}^{-1}\frac{\eta(\frac{z+1}{3})}{\eta(z)},\ \mathfrak{g}_2(z) = \frac{\eta(\frac{z+2}{3})}{\eta(z)},\ \mathfrak{g}_3(z) = \sqrt{3}\frac{\eta(3z)}{\eta(z)}.$$

Note that the functions in (2.2) are modular of level 72. For the generating matrices $S, T \in SL_2(\mathbb{Z})$ given by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, the transformation rules $\eta \circ S(z) = \sqrt{-iz}\eta(z)$ and $\eta \circ T(z) = \zeta_{24}\eta(z)$ hold. Hence

$$\begin{aligned} (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ S &= (\mathfrak{g}_3, \zeta_{24}^{-2}\mathfrak{g}_2, \zeta_{24}^{2}\mathfrak{g}_1, \mathfrak{g}_0), \\ (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ T &= (\mathfrak{g}_1, \zeta_{24}^{-2}\mathfrak{g}_2, \mathfrak{g}_0, \zeta_{24}^{2}\mathfrak{g}_3). \end{aligned} \tag{2.3}$$

3. Results

In this section, we compute the action of a primitive quadratic form $[a,-b,c]$ on the class invariant $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$. For that we need to find the action of $M_m \in GL_2(\mathbb{Z}/m\mathbb{Z})$ with $m=8,9$. Combining Lemma 6 of [2] and the transformation rule (2.3), we obtain the following:

Lemma 3.1. *The actions of* $\left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right)_m$, S_m *and* T_m $(m=8,9)$ *on* $\mathfrak{g}_i^2$ $(i=0,1,2,3)$ *are given by*

	$\mathfrak{g}_0^2$	$\mathfrak{g}_1^2$	$\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
$\left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right)_8$	$\mathfrak{g}_0^2$	$\mathfrak{g}_1^2$	$\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
S_8	$-\mathfrak{g}_0^2$	$-\mathfrak{g}_1^2$	$-\mathfrak{g}_2^2$	$-\mathfrak{g}_3^2$
T_8	$-\mathfrak{g}_0^2$	$-\mathfrak{g}_1^2$	$-\mathfrak{g}_2^2$	$-\mathfrak{g}_3^2$
$\left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right)_9$, $x=-3k+1$	$\mathfrak{g}_0^2$	$\zeta_3^{2k}\mathfrak{g}_1^2$	$\zeta_3^{k}\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
$\left(\begin{smallmatrix}1&0\\0&x\end{smallmatrix}\right)_9$, $x=-3k-1$	$\mathfrak{g}_0^2$	$\zeta_3^{2k}\mathfrak{g}_2^2$	$\zeta_3^{k}\mathfrak{g}_1^2$	$\mathfrak{g}_3^2$
S_9	$-\mathfrak{g}_3^2$	$\zeta_3\mathfrak{g}_2^2$	$\zeta_3^2\mathfrak{g}_1^2$	$-\mathfrak{g}_0^2$
T_9	$-\mathfrak{g}_1^2$	$\zeta_3\mathfrak{g}_2^2$	$-\mathfrak{g}_0^2$	$\zeta_3^2\mathfrak{g}_3^2$

Theorem 2.1 gives a matrix $M \in \mathrm{GL}_2(\mathbb{Z}/72\mathbb{Z})$ that satisfies

$$\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]} = (\zeta_3^2\zeta_4\mathfrak{g}_1^2)^M(\tau_Q).$$

Also

$$(\zeta_3^2\zeta_4\mathfrak{g}_1^2)^M = (\zeta_3^2(\zeta_4\mathfrak{g}_1^2)^{M_8})^{M_9}.$$

By Lemma 3.1, we have

$$(\zeta_4\mathfrak{g}_1^2)^{M_8} = u\mu_4\mathfrak{g}_1^2,$$

where $u=(-1)^{a+\frac{b-2}{2}}$ and $\mu_4=\zeta_4^{a+(a+1)(\frac{b^2}{4}+c)}$. Using this, together with Lemma 3.1, we have the following theorems.

Theorem 3.2. *Let* $D \equiv 64 \pmod{72}$ *be a discriminant of an order* $\mathcal{O}=[\theta,1]$ *in an imaginary quadratic field. Let* $\theta=\frac{\sqrt{D}}{2}$, $\tau_Q=\frac{-b+\sqrt{D}}{2a}$, $u=(-1)^{a+\frac{b-2}{2}}$ *and* $\mu_4=\zeta_4^{a+(a+1)(\frac{b^2}{4}+c)}$. *If* $Q=[a,b,c]$ *is a reduced primitive quadratic form with discriminant* D, *then the action of* $[a,-b,c]$ *on* $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)$ *is as follows:*

(1) *The case* $3\nmid a$.

a) *If* $b\equiv 1 \pmod 3$, *then* $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b\equiv 1 \pmod 9$	$b\equiv 4 \pmod 9$	$b\equiv 7 \pmod 9$
$a\equiv 1 \pmod 9$	$-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$
$a\equiv 2 \pmod 9$	$-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$

$a \equiv 4 \pmod 9$	$-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$-u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\mu_4\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q)$

b) *If* $a+b \equiv 1 \pmod 3$, *then* $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$a \equiv 1 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$a \equiv 2 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$

c) *If* $b \not\equiv 1 \pmod 3$ *and* $a+b \not\equiv 1 \pmod 3$, *then* $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$a \equiv 1 \pmod 9$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$a \equiv 2 \pmod 9$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$

(2) *The case* $3|a$ *and* $3 \nmid c$.

a) *If* $b \equiv 2 \pmod 3$, *then* $\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$c \equiv 1 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 2 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 4 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 5 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 7 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 8 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\mu_4\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q)$

b) *If* $b \not\equiv 2 \pmod 3$ *and* $b + c \equiv 0 \pmod 3$, *then*

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$c \equiv 2 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$
$c \equiv 5 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$
$c \equiv 8 \pmod 9$	$u\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_2^2(\tau_Q)$

c) *If* $b + c \equiv 2 \pmod 3$, *then*

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$c \equiv 1 \pmod 9$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$
$c \equiv 4 \pmod 9$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$
$c \equiv 7 \pmod 9$	$u\zeta_3^2\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\mu_4\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mu_4\mathfrak{g}_1^2(\tau_Q)$

(3) *The case* $3|a$ *and* $3|c$.

a) *If* $b \equiv 1 \pmod 9$, *then*

$$\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]} = \begin{cases} -u\mu_4\mathfrak{g}_0^2(\tau_Q) & \text{if } c \equiv 0 \pmod 9; \\ -u\zeta_3\mu_4\mathfrak{g}_0^2(\tau_Q) & \text{if } c \equiv 3 \pmod 9; \\ -u\zeta_3^2\mu_4\mathfrak{g}_0^2(\tau_Q) & \text{if } c \equiv 6 \pmod 9; \end{cases}$$

b) *If* $b \equiv 8 \pmod 9$, *then*

$$\zeta_3^2\zeta_4\mathfrak{g}_1^2(\theta)^{[a,-b,c]} = \begin{cases} u\mu_4\mathfrak{g}_3^2(\tau_Q) & \text{if } a \equiv 0 \pmod 9; \\ u\zeta_3\mu_4\mathfrak{g}_3^2(\tau_Q) & \text{if } a \equiv 3 \pmod 9; \\ u\zeta_3^2\mu_4\mathfrak{g}_3^2(\tau_Q) & \text{if } a \equiv 6 \pmod 9; \end{cases}$$

References

[1] H. H. Chan, A. Gee, and V. Tan, *Cubic singular moduli, Ramanujan's class invariants λ_n and the explicit Shimura reciprocity law*, Pacific J. Math. **208** (2003), 23–37.

[2] A. Gee, *Class invariants by Shimura's reciprocity law*, J. Théor. Nombre Bordeaux **11** (1999), 45–72.

[3] A. Gee, *Class fields by Shimura reciprocity*, Ph. D. Thesis, Universiteit van Amsterdam, 2001.

[4] A. Gee and P. Stevenhagen, *Generating class fields using Shimura reciprocity*, Proceedings of the Third International Symposium on Algorithmic Number Theory, Lecture Notes in Computer Sciences **1423** pp. 441-453, Springer-Verlag, 1998.

[5] D. Jeon, *Formulas of Galois actions of some class invariants over quartic number fields with discriminants* $D \equiv 1\,(mod\,12)$, J. of Chungcheong Math. Soc. **22** (2009), 799–814.

[6] D. Jeon, *Galois actions of a class invariant over quadratic number fields with discriminant* $D \equiv -3\,(mod\,36)$, J. of Chungcheong Math. Soc. **23** (2010), 853–860

[7] D. Jeon, *Galois actions of a class invariant over quadratic number fields with discriminant* $D \equiv 21\,(mod\,36)$, J. of Chungcheong Math. Soc. **24** (2011), 921–925

[8] G. Shimura, *Introduction to the arithmetic theory of automorphic forms*, Princeton University Press, 1971.

[9] P. Stevenhagen, 'Hilbert's 12th problem, complex multiplication and Shimura reciprocity'Class field theory-its centenary and prospect, ed. K. Miyake, *Adv. Studies in pure math.* **30** (2001), 161-176.

[10] H. Weber, *Lehrbuch der Algebra*, dritter Band, Friedrich Vieweg und Sohn, 1908.

*
Department of Mathematics Education
Kongju National University
Kongju 314-701, Republic of Korea
E-mail: dyjeon@kongju.ac.kr