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SEMI–COMPATIBILITY, COMPATIBILITY AND FIXED POINT THEOREMS IN FUZZY METRIC SPACE

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Abstract. The object of this paper is to introduce the concept of a pair of semi-compatible self-maps in a fuzzy metric space to establish a fixed point theorem for four self-maps. It offers an extension of Vasuki [10] to four self-maps under the assumption of semi-compatibility and compatibility, repsectively. At the same time, these results give the alternate results of Grebiec [5] and Vasuki [9] as well.

1. Introduction

Zadeh's [11] introduction of the notion of fuzzy set laid the foundation of fuzzy mathematics. George and Veeramani [4] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6]. Vasuki [10] and Singh and Chauhan [8] introduced the concept of R-weakly commuting and compatible maps, respectively, in fuzzy metric space. Recently, Cho et al [2] initiated the concept of compatible maps of type (β) in fuzzy metric spaces by giving interesting relationship of these type of mapping with compatible and compatible of type (α) mappings.

In [3], Cho, Sharma and Sahu introduced the non-symmetrical concept of semi-compatibility of maps in d-complete topological spaces. They defined a pair of self-maps (S,T) to be semi-compatible if the condition (i) $Sy = Ty$ implies $STy = TSy$ and (ii) $\{Sx_n\} \to x$ and

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$\{Tx_n\} \to x$ imply $STx_n \to Tx$, as $n \to \infty$, hold. However, (ii) implies (i), taking $x_n = y$ and $x = Ty = Sy$. So we define semi-compatibility by the condition (ii) only in the setting of fuzzy metric space.

In this paper, the notions of weak-compatible and semi-compatible maps in fuzzy metric space have been introduced by giving interesting relationship of this type of maps with compatible and compatible of type (α) and compatible of type (β) maps. Using these concepts, one can obtain some generalized fixed point theorem which extends the result of Vasuki [10] in the following ways:

(a) by increasing the number of self-maps from 2 to 4,

(b) by reducing the assumption of R-weakly commuting maps to that of compatible or semi-compatible and weak-compatible maps only.

2. Preliminaries

Definition 2.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for $a, b, c, d \in [0,1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. ([9]) The 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$

($F.M$-1) $M(x, y, 0) = 0$,

($F.M$-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

($F.M$-3) $M(x, y, t) = M(y, x, t)$,

($F.M$-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

($F.M$-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous,

($F.M$-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

EXAMPLE 2.1. ([4]) Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and all $t > 0$. Then $(X, M, *)$ is a fuzzy metric space. It is called the fuzzy metric space induced by the metric d.

LEMMA 2.1. ([5]) *For all* $x, y \in X$, $M(x, y, \cdot)$ *is a non-decreasing function.*

DEFINITION 2.3. ([5]) Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. Further, the sequence $\{x_n\}$ is said to be a *Cauchy sequence* if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$. The space is said to be *complete* if every Cauchy sequence in X converges to a point in X.

3. Compatible maps

In this section, we give the concept of different types of compatible maps and some properties of them for our main result.

DEFINITION 3.1. ([10]) Two maps A and S from a fuzzy metric space $(X, M, *)$ into itself are said to be *R-weakly commuting* if there exists a positive real number R such that for each $x \in X$

$$M(ASx, SAx, Rt) \geq M(Ax, Sx, t)$$

for all $t > 0$.

DEFINITION 3.2. ([7]) Two maps A and B from a fuzzy metric space $(X, M, *)$ into itself are said to be *compatible* if

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$$

for some $x \in X$.

Definition 3.3. ([1]) Two maps A and B from a fuzzy metric space $(X, M, *)$ into itself are said to be *compatible of type* (α) if

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1$$
$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$$

for some $x \in X$.

Definition 3.4. ([2]) Two maps A and B from a fuzzy metric space $(X, M, *)$ into itself are said to be *compatible of type* (β) if

$$\lim_{n\to\infty} M(A^2x_n, B^2x_n, t) = 1$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$$

for some $x \in X$.

Definition 3.5. Two maps A and B from a fuzzy metric space $(X, M, *)$ into itself are said to be *weak-compatible* if they commute at their coincidence points, i.e., $Ax = Bx$ implies $ABx = BAx$.

Definition 3.6. A pair (A, S) of self-maps of a fuzzy metric space $(X, M, *)$ is said to be *semi-compatible* if $\lim_{n\to\infty} ASx_n = Sx$ whenever $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X.$$

It follows that (A, S) is semi-compatible and $Ay = Sy$ then $ASy = SAy$.

REMARK 3.1. Let (A, S) be a pair of self-maps of a fuzzy metric space $(X, M, *)$. Then (A, S) is R-weakly commuting implies that (A, S) is compatible, which implies that (A, S) is weak-compatible. But the converse is not true. The following is an example of a pair of self-maps which is weakly compatible, but not compatible. Hence it is not R-weakly commuting.

EXAMPLE 3.1. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$, t-norm is defined by $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ for all $x, y \in X$ and all $t > 0$. Define self-maps A and S on X as follows:

$$Ax = \begin{cases} 2 - x & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x \le 2 \end{cases}$$

$$Sx = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x \le 2 \end{cases}$$

Take $x_n = 1 - \frac{1}{n}$. Then $x_n \to 1$, $x_n < 1$ and $2 - x_n > 1$ for all n. Also Ax_n, $Sx_n \to 1$ and $n \to \infty$. Now

$$M(ASx_n, SAx_n, t) = e^{-\frac{|ASx_n - SAx_n|}{t}} \to e^{-\frac{1}{t}} \neq 1$$

as $n \to \infty$. So A and S are not compatible. The set of coincident points of A and S is $[1, 2]$. For any $x \in [1, 2]$, $Ax = Sx = 2$ and $ASx = A(2) = 2 = S(2) = SAx$. Thus A and S are weak-compatible but not compatible.

PROPOSITION 3.1. *Let A and S be self-maps on a fuzzy metric space $(X, M, *)$. Assume that S is continuous. Then (A, S) is semi-compatible if and only if (A, S) is compatible.*

Proof. Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to u$ and $\{Sx_n\} \to u$. Since S is continuous, we have $SAx_n \to Su$.

Suppose that (A, S) is semi-compatible. Then

$$\lim_{n\to\infty} M(ASx_n, Su, \frac{t}{2}) = 1$$
$$\lim_{n\to\infty} M(SAx_n, Su, \frac{t}{2}) = 1.$$

Now

$$M(ASx_n, SAx_n, t) \geq M(ASx_n, Su, \frac{t}{2}) * M(SAx_n, Su, \frac{t}{2}).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1.$$

Hence the pair (A, S) is compatible.

Conversely, suppose that (A, S) be compatible. Then for all $t > 0$ we have

$$\lim_{n\to\infty} M(ASx_n, SAx_n, \frac{t}{2}) = 1$$
$$\lim_{n\to\infty} M(SAx_n, Su, \frac{t}{2}) = 1.$$

Now,

$$M(ASx_n, Su, t) \geq M(ASx_n, SAx_n, \frac{t}{2}) * M(SAx_n, Su, \frac{t}{2}).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(ASx_n, Su, t) = 1.$$

Hence $ASx_n \to Su$, i.e., (A, S) is semi-compatible. □

Proposition 3.2. *Let A and S be continuous self-maps on a fuzzy metric space $(X, M, *)$. If (A, S) is semi-compatible, then (A, S) is compatible of type (α).*

Proof. Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to u$ and $\{Sx_n\} \to u$. Since A and S are continuous, we have $A^2x_n \to Au, S^2x_n \to Su, ASx_n \to Au$ and $SAx_n \to Su$. Since (A, S) is semi-compatible, we have $ASx_n \to Su$. Since the limit of the sequence is unique, we have $Au = Su$. Thus

$$\lim_{n\to\infty} M(A^2x_n, Au, \frac{t}{2}) = 1$$
$$\lim_{n\to\infty} M(S^2x_n, Su, \frac{t}{2}) = 1$$
$$\lim_{n\to\infty} M(SAx_n, Au, \frac{t}{2}) = 1$$
$$\lim_{n\to\infty} M(ASx_n, Su, \frac{t}{2}) = 1$$

Now

$$M(A^2x_n, SAx_n, t) \geq M(A^2x_n, Au, \frac{t}{2}) * M(SAx_n, Au, \frac{t}{2}).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(A^2x_n, SAx_n, t) = 1.$$

Again

$$M(S^2x_n, ASx_n, t) \geq M(S^2x_n, Su, \frac{t}{2}) * M(ASx_n, Su, \frac{t}{2}).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(S^2x_n, ASx_n, t) = 1.$$

Thus the pair (A, S) is compatible of type (α). □

Proposition 3.3. *Let A and S be self-maps on a fuzzy metric space $(X, M, *)$. If S is continuous and (A, S) is compatible of type (α), then (A, S) is semi-compatible.*

Proof. Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to u$ and $\{Sx_n\} \to u$. Since S is continuous, we have $S^2x_n \to Su$. Since (A, S) is compatible of type (α), we have $M(S^2x_n, ASx_n, t) \to 1$. Thus for all $t > 0$

$$\lim_{n\to\infty} M(S^2x_n, Su, \frac{t}{2}) = 1$$
$$\lim_{n\to\infty} M(S^2x_n, ASx_n, \frac{t}{2}) = 1.$$

Now

$$M(ASx_n, Su, t) \geq M(ASx_n, S^2x_n, \frac{t}{2}) * M(S^2x_n, Su, \frac{t}{2}).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(ASx_n, Su, t) = 1.$$

Thus $ASx_n \to Su$ and the pair (A, S) is semi-compatible. □

Proposition 3.4. *Let A and S be continuous self-maps on a fuzzy metric space $(X, M, *)$. Then (A, S) is semi-compatible if and only if (A, S) is compatible of type (β).*

Proof. Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to u$ and $\{Sx_n\} \to u$. Since A and S are continuous, we have $A^2x_n \to Au$, $S^2x_n \to Su$ and $ASx_n \to Au$. Thus for all $t > 0$

$$\lim_{n\to\infty} M(S^2x_n, ASx_n, \frac{t}{2}) = 1$$
$$\lim_{n\to\infty} M(S^2x_n, Su, \frac{t}{2}) = 1.$$

Suppose (A, S) is semi-compatible. Then $ASx_n \to Su$. So $Au = Su$.

$$\begin{aligned} M(A^2x_n, S^2x_n, t) &\geq M(A^2x_n, Au, \frac{t}{2}) * M(S^2x_n, Au, \frac{t}{2}) \\ &= M(A^2x_n, Au, \frac{t}{2}) * M(S^2x_n, Su, \frac{t}{2}). \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(A^2x_n, S^2x_n, t) = 1.$$

Thus (A, S) is compatible of type (β).

Conversely, suppose (A, S) is compatible of type (β). Then we have

$$\lim_{n\to\infty} M(A^2x_n, S^2x_n, \frac{t}{4}) = 1.$$

Now

$$\begin{aligned} M(Au,Su,t) &\geq M(Au, A^2x_n, \frac{t}{2}) * M(A^2x_n, Su, \frac{t}{2}) \\ &\geq M(Au, A^2x_n, \frac{t}{2}) * M(A^2x_n, S^2x_n, \frac{t}{4}) * M(S^2x_n, Su, \frac{t}{4}). \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(Au, Su, t) = 1$$

for all $t > 0$. Thus $Au = Su$. Now $ASx_n \to Au$. So (A, S) is semi-compatible. □

The following is an example of a pair (S, T) of self-maps, which is semi-compatible, but not compatible. Further, it is shown that the semi-compatibility of the pair (S, T) need not imply the semi-compatibility of (T, S).

Example 3.2. Let $X = [0,1]$ and (X, M, t) be the induced fuzzy metric space with $M(x, y, t) = \frac{t}{t+|x-y|}$. Define a self-map S on X as follows:

$$Sx = \begin{cases} x & \text{if } 0 \leq x < \dfrac{1}{2} \\ 1 & \text{if } x \geq \dfrac{1}{2} \end{cases}$$

Let I be the identity map on X and $x_n = \frac{1}{2} - \frac{1}{n}$. Then $\{Ix_n\} = \{x_n\} \to \frac{1}{2}$ and $\{Sx_n\} \to \frac{1}{2} \neq S(\frac{1}{2})$. Thus (I, S) is not semi-compatible though it is compatible. For a sequence $\{x_n\}$ in X such that $\{x_n\} \to x$ and $\{Sx_n\} \to x$, we have $\{SIx_n\} = \{Sx_n\} \to x = Ix$. Thus (S, I) is semi-compatible.

Remark 3.2. The above example gives an important aspect of semi-compatibility as the pair (I, S) is commuting, weakly commuting, compatible, and weak-compatible, but it is not semi-compatible.

Example 3.3. Let $(X, M, *)$ be the fuzzy metric space as defined in Example 3.1. Define self-maps A and S on X as follows:

$$Ax = \begin{cases} 2 & \text{if } 0 \leq x \leq 1 \\ \dfrac{x}{2} & \text{if } 1 < x \leq 2 \end{cases}$$

$$Sx = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 2 & \text{if } x = 1 \\ \dfrac{x+3}{5} & \text{if } 1 < x \leq 2 \end{cases}$$

and $x_n = 2 - \frac{1}{2n}$. Then we have $S(1) = A(1) = 2$ and $S(2) = A(2) = 1$. $SA(1) = AS(1) = 1$ and $SA(2) = AS(2) = 2$. Hence $Ax_n \to 1$ and $Sx_n \to 2$ and $SAx_n \to 1$ as $n \to \infty$.

Now

$$\lim_{n\to\infty} M(ASx_n, Sy, t) = M(2, 2, t) = 1$$

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = M(2, 1, t) = \frac{t}{1+t} < 1.$$

Hence (A, S) is semi-compatible but not compatible.

In [10], Vasuki proved the following theorem for R-weakly commuting pair of self-maps.

THEOREM 3.5. ([10]) *Let f and g be R-weakly commuting self-maps on a complete fuzzy metric space $(X, M, *)$ such that*

$$M(fx, fy, t) \geq r(M(gx, gy, t)),$$

where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. If $f(X) \subset g(X)$ and either f or g is continuous then f and g have a unique common fixed point.

4. Main results

THEOREM 4.1. *Let A, B, S and T be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying*

(1) $A(X) \subset T(X), B(X) \subset S(X)$,

(2) *one of A and B is continuous,*

(3) *(A, S) is semi-compatible and (B, T) is weak-compatible,*

(4) *for all $x, y \in X$ and $t > 0$*

$$M(Ax, By, t) \geq r(M(Sx, Ty, t)),$$

where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. Then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point for which there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \cdots$. Using (4) with $x = x_{2n}, y = x_{2n+1}$, we get

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \geq r(M(Sx_{2n}, Tx_{2n+1}, t)) \\ &= r(M(y_{2n}, y_{2n+1}, t)) > M(y_{2n}, y_{2n+1}, t). \end{aligned}$$

Similarly,

$$M(y_{2n+2}, y_{2n+3}, t) > M(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$M(y_{n+1}, y_n, t) > r(M(y_n, y_{n-1}, t)) > M(y_n, y_{n-1}, t).$$

Thus $\{M(y_{n+1}, y_n, t)\}$ is an increasing sequence of positive real numbers in $[0, 1]$, and tends to a limit $l \leq 1$. If $l < 1$, then

$$\lim_{n\to\infty} M(y_{n+1}, y_n, t) = l > r(l) > l,$$

which is a contradiction. So $l = 1$.

Now for any positive integer p

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, \frac{t}{p}) * M(y_{n+1}, y_{n+2}, \frac{t}{p}) * \cdots * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}).$$

Taking limit as $n \to \infty$,

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \cdots * 1 = 1.$$

So

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1.$$

Thus $\{y_n\}$ is a Cauchy sequence in X. By the completeness of X, $\{y_n\}$ converges to $z \in X$. Hence

$$Ax_{2n} \to z, \quad Sx_{2n} \to z, \quad Tx_{2n+1} \to z, \quad Bx_{2n+1} \to z. \tag{5}$$

Case when A is continuous

Since A is continuous and (A, S) is semi-compatible, we get

$$ASx_{2n} \to Az \quad \& \quad ASx_{2n} \to Sz. \tag{6}$$

Since the limit in fuzzy metric space is unique, we get

$$Az = Sz. \tag{7}$$

Step I. We prove $Az = z$. Put $x = z$, $y = x_{2n+1}$ in (4) and let $Az \neq z$. Then

$$M(Az, Bx_{2n+1}, t) \geq r(M(Sz, Tx_{2n+1}, t)) > M(Sz, Tx_{2n+1}, t).$$

Taking limit as $n \to \infty$ and using (5) and (7), we get

$$M(Az, z, t) \geq r(M(Az, z, t)) > M(Az, z, t),$$

which is a contradiction and hence $z = Az = Sz$.

Step II. Since $A(X) \subset T(X)$, there exists $u \in X$ such that $z = Az = Tu$. Put $x = x_{2n}, y = u$ in (4), we get

$$M(Ax_{2n}, Bu, t) \geq r(M(Sx_{2n}, Tu, t)).$$

Taking limit as $n \to \infty$ and using (5), we get

$$M(z, Bu, t) \geq r(M(z, z, t)) = r(1) = 1,$$

which gives $z = Bu = Tu$ and the weak-compatibility of (B, T) gives $TBu = BTu$, i.e., $Tz = Bz$.

Step III. Putting $x = z, y = z$ in (4) and assuming $Az \neq Bz$, we get

$$\begin{aligned} M(Az, Bz, t) &\geq r(M(Sz, Tz, t)) = r(M(Az, Bz, t)) \\ &> M(Az, Bzt), \end{aligned}$$

which is a contradiction, and we get $Az = Bz = z$. Combining all the results, we get

$$z = Az = Bz = Sz = Tz,$$

i.e., z is a common fixed point of A, B, S and T.

Case when S is continuous

Since S is continuous and (A, S) is semi-compatible, we get

$$SAx_{2n} \to Sz, \quad S^2x_{2n} \to Sz, \quad ASx_{2n} \to Sz. \tag{8}$$

Thus

$$\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} ASx_{2n} = Sz.$$

Now we prove $Sz = z$. Putting $x = Sx_{2n}$, $y = x_{2n+1}$ in (4) and assuming $Sz \neq z$, we get

$$M(ASx_{2n}, Bx_{2n+1}, t) \geq r(M(SSx_{2n}, Tx_{2n+1}, t)).$$

Taking limit as $n \to \infty$ and using (5) and (8), we get

$$M(Sz, z, t) \geq r(M(Sz, z, t)) > M(Sz, z, t),$$

which is a contradiction and thus $Sz = z$.

Put $x = z, y = x_{2n+1}$ in (4). Then we get

$$M(Az, Bx_{2n+1}, t) \geq r(M(Sz, Tx_{2n+1}, t)).$$

Taking limit as $n \to \infty$ and using (5), we get

$$M(Az, z, t) \geq r(M(z, z, t)) = r(1) = 1,$$

which gives $z = Az$, and hence $Sz = z = Az$.

Also, it follows from Steps II and III that $Bz = Tz = z$. Hence we get

$$z = Az = Bz = Sz = Tz.$$

So z is a common fixed point of A, B, S and T.

Uniqueness

Let z_1 be another common fixed point of A, B, S and T. Then $z_1 = Az_1 = Bz_1 = Sz_1 = Tz_1$ and $z = Az = Bz = Sz = Tz$. Assuming $z \neq z_1$ and using (4), we get

$$\begin{aligned} M(z, z_1, t) &= M(Az, Bz_1, t) \geq r(M(Sz, Tz_1, t)) \\ &= r(M(z, z_1, t)) > M(z, z_1, t), \end{aligned}$$

which is a contradiction. Hence $z = z_1$ and so z is the unique common fixed point of A, B, S and T. □

COROLLARY 4.2. *Let A, B, S and T be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying* (1), (4) *and*

(9) (A, S) *and* (B, T) *are semi-compatible,*

(10) *one of A, B, S and T is continuous.*

Then A, B, S and T have a unique common fixed point.

Proof. Since (B, T) is semi-compatible, we get (B, T) is weak-compatible, etc. And the result follows from Theorem 4.1. □

If we take $A = B = f$ and $S = T = g$ in Theorem 4.1, then we get the following.

THEOREM 4.3. *Let $(X, M, *)$ be a complete fuzzy metric space, and let f and g be semi-compatible self-maps on X satisfying the condition:*

$$M(fx, fy, t) \geq r(M(gx, gy, t)),$$

where $r : [0, 1] \to [0, 1]$ is a continuous function such that $t(t) > t$ for each $0 < t < 1$. If $f(X) \subset g(X)$ and either f or g is continuous, then f and g have a unique common fixed point.

REMARK 4.1. This result proves that the theorem of Vasuki [10] holds well even if the pair (f, g) is semi-compatible.

Take $S = I$ in Theorem 4.1. We have the following result for three self-maps, none of which is continuous and just a pair of them is needed to be weak-compatible only.

Corollary 4.4. *Let A, B and T be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying*

(11) $A(X) \subset T(X)$,

(12) (B, T) *is weak-compatible,*

(13) *for all $x, y \in X$ and $t > 0$*

$$M(Ax, By, t) \geq r(M(x, Ty, t)),$$

where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. Then A, B and T have a unique common fixed point.

Again if we take $S = T = I$ in Theorem 4.1 then the conditions (1), (2) and (3) are satisfied trivially and we get the following important result to be used for a unique common fixed point of a sequence of self-maps.

Corollary 4.5. *Let A and B be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying*

$$M(Ax, By, t) \geq r(M(x, y, t))$$

for all $x, y \in X$, where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. Then A and B have a unique common fixed point.

In Grebiek [5], the following version of Banach contraction theorem has been established for fuzzy metric space.

Theorem 4.6. ([5]) *Let $(X, M, *)$ be a complete fuzzy metric space where $*$ is a continuous t-norm and T a self-map on X such that*

$$M(Tx, Ty, t) \geq M(x, y, t)$$

for all $x, y \in X$ and $t > 0$. Then T has a unique fixed point.

REMARK 4.2. If we take $A = B = T$ in Corollary 4.5, then we have an alternate result of the above result of [5].

THEOREM 4.7. ([9]) *Let $\{T_n\}$ be a sequence of self-maps on a complete fuzzy metric space $(X, M, *)$, where $*$ is a continuous t-norm, such that for any two maps T_i and T_j, we have*

$$M(T_i^m x, T_j^m y, \alpha_{i,j} t) \geq M(x, y, t)$$

for all $x, y \in X$ and some m and $0 < \alpha_{i,j} < 1$, $i, j = 1, 2, \cdots$. Then $\{T_n\}$ has a unique common fixed point.

The following is an alternate result of it.

THEOREM 4.8. *Let $\{A_n\}$ be a sequence of self-maps on a complete fuzzy metric space $(X, M, *)$ such that every pair of consecutive maps satisfies*

$$M(A_i^{m_i} x, A_{i+1}^{m_{i+1}} y, t) \geq r_i(M(x, y, t))$$

for all $x, y \in X$, $t > 0$ and $r_i : [0, 1] \to [0, 1]$ are continuous functions such that $r_i(t) > t$ for each $0 < t < 1$. Then $\{A_n\}$ has a unique common fixed point.

Proof. By Corollary 4.5, the pair $(A_i^{m_i}, A_{i+1}^{m_{i+1}})$ has a unique common fixed point, say, u. Hence $u = A_i^{m_i} u = A_{i+1}^{m_{i+1}} u$. Now $A_i^{m_i}(A_i u) = A_i(A_i^{m_i} u) = A_i u$, i.e., $A_i u$ is a fixed point of $A_i^{m_i}$. Similarly, $A_{i+1} u$ is a fixed point of $A_{i+1}^{m_{i+1}}$. Putting $x = A_i u$ and $y = u$ in the above condition, we get

$$M(A_i^{m_i} A_i u, A_{i+1}^{m_{i+1}} u, r_i(t)) \geq r_i(M(A_i u, u, t))$$

implies

$$M(A_i u, u, t) \geq r_i(M(A_i u, u, t)),$$

which gives $A_i u = u$. Similarly, we show that $A_{i+1}u = u$. Thus $A_i u = A_{i+1}u = u$. Therefore, u is a common fixed point of A_i and A_{i+1}. If v is another common fixed point of A_i and A_{i+1}, then v is a common fixed point of $A_i^{m_i}$ and $A_{i+1}^{m_{i+1}}$, which is unique. Hence $u = v$. Thus every pair of two consecutive maps has a unique common fixed point. Let u_1 be the common fixed point of the pair (A_1, A_2) and u_2 that of the pair (A_2, A_3). Putting $x = u_1, y = u_2$ in the given contraction condition taking $i = 1$, we get

$$M(u_1, u_2, t) \geq r_1(M(u_1, u_2, t)),$$

which implies $u_1 = u_2$. Thus each consecutive pair of $\{A_n\}$ has the same unique common fixed point, which must be the unique common fixed point of $\{A_n\}$. □

THEOREM 4.9. *Let A, B, S and T be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying* (1), (2), (4) *and*

(14) (A, S) *is compatible and* (B, T) *is weak-compatible.*

Then A, B, S and T have a unique common fixed point.

Proof. In view of Proposition 3.1 and Theorem 4.1, it suffices to prove the theorem when A is continuous. As in the proof of Theorem 4.1, construct a sequence $\{y_n\}$ which is a Cauchy sequence in X and hence it converges to some $z \in X$ and (1) is true. Since A is continuous and (A, S) is compatible, we get

$$ASx_{2n} \to Az, \quad A^2x_{2n} \to Sz, \quad SAx_{2n} \to Az. \tag{15}$$

Step I. We now prove $Az = z$. Put $x = Ax_{2n}, y = x_{2n+1}$ in (4) and assume that $Az \neq z$. Then

$$\begin{aligned} M(AAx_{2n}, Bx_{2n+1}, t) &\geq r(M(SAx_{2n}, Tx_{2n+1}, t)) \\ &> M(SAx_{2n}, Tx_{2n+1}, t). \end{aligned}$$

Taking limit as $n \to \infty$ and using (15) and (5), we get

$$M(Az, z, t) > M(Az, z, t),$$

which is a contradiction. Hence $z = Az$.

Step II. Since $A(X) \subset T(X)$, there exists $u \in X$ such that $z = Az = Tu$. Putting $x = x_{2n}, y = u$ in (4), we have

$$M(Ax_{2n}, Bu, t) \geq r(M(Sx_{2n}, u, t)).$$

Taking limit as $n \to \infty$ and using (5), we get

$$M(z, Bu, t) \geq r(M(z, z, t)) = r(1) = 1.$$

Thus $z = Bu = Tu$. Since (B, T) is weak-compatible, we get $TBu = BTu$, i.e., $Tz = Bz$.

Step III. Since $z = Bu$ and $B(X) \subset S(X)$, there exists $v \in X$ such that $z = Bu = Sv$. Putting $x = v, y = u$ in (4), we get

$$M(Av, Bu, t) \geq r(M(Sv, Tu, t)) = r(M(z, z, t)) = r(1) = 1.$$

Thus $Av = Bu$ and hence $z = Sv = Av$. Since (A, S) is semi-compatible, we get $ASv = SAv$ and $Az = Sz = z$.

Step IV. Putting $x = z, y = z$ in (4) and assuming $Az \neq Bz$, we get

$$\begin{aligned} M(Az, Bz, t) &\geq r(M(Sz, Tz, t)) = r(M(Az, Bz, t)) \\ &> M(Az, Bz, t), \end{aligned}$$

which is a contradiction. So we get $Az = Bz = z$.

Combining all the results, we get $z = Az = Bz = Sz = Tz$, i.e., z is a common fixed point of A, B, S and T, and the uniqueness follows as in the proof of Theorem 4.1. $\square$

Corollary 4.10. *Let A, B, S and T be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying* (1), (4) *and*

(16) *(A, S) and (B, T) are compatible,*

(17) *one of A, B, S and T is continuous.*

Then A, B, S and T have a unique common fixed point.

Proof. Since compatibility implies weak-compatibility, the proof follows from Theorem 4.9. □

If we take $A = B = f$ and $S = T = g$ in Theorem 4.9, we get the following.

Theorem 4.11. *Let f and g be compatible self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying*

$$M(fx, fy, t) \geq r(M(gx, gy, t)),$$

where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$. If $f(X) \subset g(X)$ and either f or g is continuous, then f and g have a unique common fixed point.

Remark 4.3. Theorem 4.11 generalizes Theorem of Vasuki [10] by assuming only compatibility of the pair (f, g) in place of its being R-weakly commuting. Thus Theorem 4.9 is a still better generalization of a result of [10] for four self-maps.

Corollary 4.12. *Let A, B, S and T be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying* (1), (4) *and*

(18) *(A, S) is compatible of type (α) and (B, T) is weak-compatible,*

(19) *S is continuous.*

Then A, B, S and T have a unique common fixed point.

Proof. The proof follows from Theorem 4.1 and Proposition 3.3. □

COROLLARY 4.13. *Let A, B, S and T be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying* (1), (4) *and*

(20) *(A, S) is compatible of type (β) and (B, T) is weak-compatible,*

(21) *A and S are continuous.*

Then A, B, S and T have a unique common fixed point.

Proof. The proof follows from Theorem 4.1 and Proposition 3.4. □

Taking $A = I$ in Theorem 4.8, we have another result for three self-maps, none of which are continuous and just a pair of them is needed to be weak-compatible only.

COROLLARY 4.14. *Let B, S and T be self-maps on a complete fuzzy metric space $(X, M, *)$ satisfying*

(22) *$B(X) \subset S(X)$ and T is surjective,*

(23) *(B, T) is weak-compatible,*

(24) *for all $x, y \in X$ and $t > 0$,*

$$M(x, By, t) \geq r(M(Sx, Ty, t)),$$

where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$.

Then B, S and T have a unique common fixed point.

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