JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 18, No. 2, October 2005

FINITE GROUP ACTIONS ON THE 3–DIMENSIONAL NILMANIFOLD

Daehwan Goo* and Joonkook Shin**

Abstract. We study only free actions of finite groups G on the 3-dimensional nilmanifold, up to topological conjugacy which yields an infra-nilmanifold of type 2.

1. Introduction

Free actions of finite, cyclic and abelian groups on the 3-torus were studied in [4], [7] and [8], respectively. It is known [3, Proposition 6.1.] that there are 15 classes of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant. It is interesting that if a finite group acts freely on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2$, then it is cyclic [2]. Free actions of finite abelian groups on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$ were classified in [1]. All these works can be understood easily by the works of Bieberbach, L. Auslander and Waldhausen [5, 6, 10]. In this paper we study free actions of finite groups on the 3-dimensional nilmanifolds yielding an infra-nilmanifold of type 2 which includes Theorem 3.3 of [1] as corollary. We shall use all notations and most of the Introduction and Section 2 of [1] in our Introduction and Section 2.

Let $\mathcal{H}$ be the 3–dimensional Heisenberg group; i.e., $\mathcal{H}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. Thus

Received by the editors on August 25, 2005.
2000 *Mathematics Subject Classifications*: Primary 57S25, Secondary 57S17.
Key words and phrases: group actions, Heisenberg group, almost Bieberbach groups, affine conjugacy.

$\mathcal{H}$ is a simply connected, 2-step nilpotent Lie group, and it fits an exact sequence

$$1 \to \mathbb{R} \to \mathcal{H} \to \mathbb{R}^2 \to 1,$$

where $\mathbb{R} = \mathcal{Z}(\mathcal{H})$, the center of $\mathcal{H}$. Hence $\mathcal{H}$ has the structure of a line bundle over $\mathbb{R}^2$. We take a left invariant metric coming from the orthonormal basis

$$\left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

for the Lie algebra of $\mathcal{H}$. This is, what is called, the Nil-geometry and its isometry group is $\mathrm{Isom}(\mathcal{H}) = \mathcal{H} \rtimes O(2)$ [9]. All isometries of $\mathcal{H}$ preserve orientation and the bundle structure.

We say that a closed 3-dimensional manifold M has a Nil-geometry if there is a subgroup π of $\mathrm{Isom}(\mathcal{H})$ so that π acts properly discontinuously and freely with quotient $M = \mathcal{H}/\pi$. The simplest such a manifold is the quotient of $\mathcal{H}$ by the lattice consisting of integral matrices. For each integer $p > 0$, let

$$\Gamma_p = \left\{ \begin{bmatrix} 1 & l & \frac{n}{p} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \middle| \; l, m, n \in \mathbb{Z} \right\}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{H}$ consisting of all integral matrices and Γ_p is a lattice of $\mathcal{H}$ containing Γ_1 with index p. Clearly

$$\mathrm{H}_1(\mathcal{H}/\Gamma_p; \mathbb{Z}) = \Gamma_p/[\Gamma_p, \Gamma_p] = \mathbb{Z}^2 \oplus \mathbb{Z}_p.$$

Note that these ${\Gamma_p}'$s produce infinitely many distinct nilmanifolds $\mathcal{N}_p = \mathcal{H}/\Gamma_p$ covered by $\mathcal{N}_1$. In this paper, we shall find all possible finite groups acting freely on each $\mathcal{N}_p$ yielding an infra-nilmanifold of type 2 in [1].

Definition 1.1. *Let groups G_i act on manifolds M_i, for $i = 1, 2$. The action (G_1, M_1) is topologically conjugate to (G_2, M_2) if there exists an isomorphism $\theta\colon G_1 \to G_2$ and a homeomorphism $h\colon M_1 \to M_2$ such that $h(g \cdot x) = \theta(g) \cdot h(x)$ for all $x \in M_1$ and all $g \in G_1$.*

2. Free actions of finite groups G on the nilmanifold

In this section, we shall find all possible finite groups acting freely (up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi$, where

$$\pi = \langle\, t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{2n},\ [t_3, \alpha] = [t_3, t_1] = [t_3, t_2] = 1, \\ \alpha^2 = t_3,\ \alpha t_1 \alpha^{-1} = t_1^{-1},\ \alpha t_2 \alpha^{-1} = t_2^{-1}\,\rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{4n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right).$$

This, as in other parts of calculations, was done by the program MATHEMATICA[11] and hand-checked.

The following proposition[1, Proposition 3.1.] is a working criterion for determining all normal nilpotent subgroups of π isomorphic to Γ_p.

Proposition 2.1. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π and isomorphic to Γ_p. Then N can be represented by a set of generators*

$$N = \langle\, t_1^{d_1} t_2^{m} t_3^{n_1},\ t_2^{d_2} t_3^{n_2},\ t_3^{\frac{2nd_1d_2}{p}}\,\rangle,$$

where d_1, d_2 are divisors of p; $0 \le m < d_2$, $0 \le n_i < \frac{2nd_1d_2}{p}$ $(i = 1, 2)$.

Lemma 2.2. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π and isomorphic to Γ_p. Then N can be represented by one of the following sets of generators*

$N_1 = \langle t_1^{d_1} t_2^{m},\ t_2^{d_2},\ t_3^{\frac{2nd_1d_2}{p}} \rangle$,

$N_2 = \langle t_1^{d_1} t_2^{m},\ t_2^{d_2} t_3^{\frac{nd_1d_2}{p}},\ t_3^{\frac{2nd_1d_2}{p}} \rangle$,

$$N_3 = \langle t_1^{d_1} t_2^m t_3^{\frac{nd_1d_2}{p}},\ t_2^{d_2},\ t_3^{\frac{2nd_1d_2}{p}} \rangle,$$

$$N_4 = \langle t_1^{d_1} t_2^m t_3^{\frac{nd_1d_2}{p}},\ t_2^{d_2} t_3^{\frac{nd_1d_2}{p}},\ t_3^{\frac{2nd_1d_2}{p}} \rangle,$$

where $d_1,\ d_2$ are divisors of p; $0 \le m < \bar{d} = (d_1, d_2)$, $\frac{pm}{d_1d_2} \in \mathbb{Z}$.

Proof. Let N be a normal nilpotent subgroup of π isomorphic to Γ_p. Then by Proposition 2.1, we have

$$N = \langle\, t_1^{d_1} t_2^m t_3^{\ell},\ t_2^{d_2} t_3^r,\ t_3^{\frac{2nd_1d_2}{p}} \,\rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{2nd_1d_2}{p}\right).$$

Remark that we obtained the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi)$ of π in [2]:

$$N_{\mathrm{Aff}(\mathcal{H})}(\pi) = \left\{ \left(\begin{bmatrix} 1 & \frac{r}{2} & * \\ 0 & 1 & \frac{s}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix},\ T \right) \right) \ \middle|\ T \in \mathrm{GL}(2, \mathbb{Z}) \right\},$$

where $r,\ s \in \mathbb{Z}$.

Let $\bar{d}$ be the greatest common divisor of d_1 and d_2, denoted by $\bar{d} = (d_1,\ d_2)$. Then there exist $s, w \in \mathbb{Z}$ such that $m = \bar{d}s + w$, where $0 \le w < \bar{d}$. It is not hard to see

$$N \sim \langle\, t_1^{d_1} t_2^w t_3^{\ell'},\ t_2^{d_2} t_3^r,\ t_3^{\frac{2nd_1d_2}{p}} \,\rangle$$

by using $N_{\mathrm{Aff}(\mathcal{H})}(\pi)$.

Since N is a normal subgroup of π, the following two relations

$$\alpha(t_1^{d_1} t_2^m t_3^{\ell})\alpha^{-1} = t_1^{-d_1} t_2^{-m} t_3^{\ell} = (t_1^{d_1} t_2^m t_3^{\ell})^{-1} t_3^{\frac{2nd_1d_2}{p}x} \ \in N,$$

$$\alpha(t_2^{d_2} t_3^r)\alpha^{-1} = t_2^{-d_2} t_3^r = (t_2^{d_2} t_3^r)^{-1} t_3^{2r} = (t_2^{d_2} t_3^r)^{-1} t_3^{\frac{2nd_1d_2}{p}\,\frac{pr}{nd_1d_2}} \ \in N,$$

show that

$$x = \frac{pl}{nd_1d_2} - \frac{pm}{d_2} \in \mathbb{Z}, \quad \frac{pr}{nd_1d_2} \in \mathbb{Z}.$$

Since $0 \leq \ell,\ r < \frac{2nd_1d_2}{p}$ by Proposition 2.1, we have $\ell = 0$ or $\frac{nd_1d_2}{p}$, and $r = 0$ or $\frac{nd_1d_2}{p}$.

Note that the relation

$$t_1(t_1^{d_1}t_2^m t_3^\ell)t_1^{-1} = (t_1^{d_1}t_2^m t_3^\ell)t_3^{\frac{2nd_1d_2}{p}(-\frac{pm}{d_1d_2})} \in N$$

shows $\frac{pm}{d_1d_2} \in \mathbb{Z}$. Therefore we have proved the lemma. □

Theorem 2.3. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π and isomorphic to Γ_p. Then there exist q affinely non-conjugate normal nilpotent subgroups N^j $(j = 0, 1, \cdots, q-1)$ of Γ_2, where $q = (\bar{d}, k)$, k can be obtained from $p = kD$ and D is the least common multiple of d_1 and d_2.*

Proof. Let N and N' be normal nilpotent subgroups of π isomorphic to Γ_p whose sets of generators are

$$N = \langle {t_1}^{d_1}t_2^m t_3^\ell,\ t_2^{d_2}t_3^r,\ t_3^{\frac{2nd_1d_2}{p}}\rangle,$$
$$N' = \langle {t_1}^{d_1}t_2^{m'} t_3^{\ell'},\ t_2^{d_2}t_3^{r'},\ t_3^{\frac{2nd_1d_2}{p}}\rangle,$$

where $0 \leq m, m' < \bar{d},\ 0 \leq \ell, r, \ell', r' < \frac{2nd_1d_2}{p}$ by Lemma 2.2. We want to show that if $m \neq m'$, then N is not affinely conjugate to N'. Assume that $N \sim N'$. Then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$$

satisfying either

(∗) $\mu({t_1}^{d_1}{t_2}^m t_3^\ell)\mu^{-1} = t_1^{d_1}{t_2}^{m'}t_3^{\ell'},\quad \mu(t_2^{d_2}t_3^r)\mu^{-1} = t_2^{d_2}t_3^{r'},$

or (∗∗) $\mu({t_1}^{d_1}{t_2}^m t_3^\ell)\mu^{-1} = t_2^{d_2}{t_3}^{r'},\quad \mu(t_2^{d_2}t_3^r)\mu^{-1} = t_1^{d_1}{t_2}^{m'}t_3^{\ell'}.$

From (∗), we obtain that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, and $cd_1 = m' - m \neq 0$. Since $d_1 \leq |c|d_1 = |m' - m| < \bar{d} \leq d_1$, we have a contradiction. However in (∗∗), we obtain the following relations

$$bd_2 = d_1,\ \ dd_2 = m',\ \ ad_1 + bm = 0,\ \ cd_1 + dm = d_2.$$

The relation $dd_2 = m' < d_2$ induces $d = 0$, $m' = 0$ and $bc = 1$. Therefore the relation $|a|d_1 = |m| < \bar{d} \le d_1$ implies $m = 0$, which is a contradiction. To complete the proof, we show that there exist q affinely non-conjugate normal nilpotent subgroups N^j $(j = 0, 1, \cdots, q-1)$. By Lemma 2.2, we know that $\frac{pm}{d_1 d_2} \in \mathbb{Z}$. Since $\bar{d} = (d_1, d_2)$ and $p = kD$, where D is the least common multiple of d_1 and d_2. we have $\frac{pm}{d_1 d_2} \in \mathbb{Z}$ if and only if $\frac{km}{\bar{d}} \in \mathbb{Z}$. Let $q = (\bar{d}, k)$. Then $\frac{km}{\bar{d}} \in \mathbb{Z}$ if and only if $\frac{k'm}{\bar{d}'} \in \mathbb{Z}$, where $k = qk'$, $\bar{d} = q\bar{d}'$, $(k', \bar{d}') = 1$. Therefore $\bar{d}'$ is a divisor of m. Since $0 \le m < \bar{d} = q\bar{d}'$, we can get $m = 0, \bar{d}', \cdots, (q-1)\bar{d}'$. □

In the following theorem, we shall show when affine conjugacy occurs between 4 types of normal nilpotent subgroups $N_i (i = 1, 2, 3, 4)$ in Lemma 2.2.

Theorem 2.4. *Let N_i $(i = 1, 2, 3, 4)$ be a normal nilpotent subgroup of π and isomorphic to Γ_p. Then we have the following:*

(1) $N_1 \sim N_2$ if and only if $m = 0$, $d_1 = p$.

(2) $N_1 \sim N_3$ if and only if $d_2 = p$.

(3) $N_1 \sim N_4$ if and only if $m = 0$, $d_1 = d_2 = p$.

(4) $N_2 \sim N_3$ if and only if $m = 0$, $d_1 = d_2$.

(5) $N_2 \sim N_4$ if and only if $d_2 = p$.

(6) $N_3 \sim N_4$ if and only if $m = 0$, $d_1 = p$.

Proof. (1) Suppose that N_1 is affinely conjugate to N_2. Then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$$

satisfying either

$$(*) \qquad \mu({t_1}^{d_1} {t_2}^{m})\mu^{-1} = t_1^{d_1} {t_2}^{m}, \quad \mu(t_2^{d_2})\mu^{-1} = t_2^{d_2} t_3^{\frac{n d_1 d_2}{p}},$$

or

$$(**) \qquad \mu({t_1}^{d_1}{t_2}^m)\mu^{-1} = t_2^{d_2} t_3^{\frac{nd_1d_2}{p}}, \quad \mu(t_2^{d_2})\mu^{-1} = t_1^{d_1}{t_2}^m.$$

From $(*)$, we obtain that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, x = -\frac{d_1}{2p}, \text{ and } y = -\frac{m}{2p}.$$

Since $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$, we have $2x = -\frac{d_1}{p} \in \mathbb{Z}$ and $2y = -\frac{m}{p} \in \mathbb{Z}$. Note that d_1, d_2 are divisors of p and $0 \leq m < d$ by Lemma 2.2. Thus we have $d_1 = p$ and $m = 0$. Similar calculations in $(**)$ induce $d_1 = d_2 = p$ and $m = 0$. Conversely, suppose that $d_1 = p$ and $m = 0$. Then $N_1 \sim N_2$ by using

$$\mu = \left(\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

(4) Suppose that N_2 is affinely conjugate to N_3. Note that

$$N_3 \sim \langle\, t_1^{d_1} t_2^m t_3^{-\frac{nd_1d_2}{p}},\ t_2^{d_2},\ t_3^{-\frac{2nd_1d_2}{p}} \,\rangle.$$

Then there exists

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$$

satisfying either

$$(*) \qquad \mu({t_1}^{d_1}{t_2}^m)\mu^{-1} = t_1^{d_1}{t_2}^m t_3^{\frac{nd_1d_2}{p}}, \quad \mu(t_2^{d_2} t_3^{\frac{nd_1d_2}{p}})\mu^{-1} = t_2^{d_2},$$

or

$$(**) \qquad \mu({t_1}^{d_1}{t_2}^m)\mu^{-1} = t_2^{d_2}, \quad \mu(t_2^{d_2} t_3^{\frac{nd_1d_2}{p}})\mu^{-1} = t_1^{d_1}{t_2}^m t_3^{-\frac{nd_1d_2}{p}}.$$

From $(*)$, we obtain $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x = \frac{d_1}{2p}$ and $y = \frac{m+d_2}{2p}$. Since $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$, we have $2x = \frac{d_1}{p} \in \mathbb{Z}$ and $2y = \frac{m+d_2}{p} \in \mathbb{Z}$.

Note that d_1, d_2 are divisors of p and $0 \le m < d_2$. Thus we have $d_1 = d_2 = p$ and $m = 0$. However in $(**)$, a similar calculation shows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $d_1 = d_2$ and $m = 0$. The converse is easy by using

$$\mu = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

The other cases can be done similarly. □

Note that π/N is abelian if and only if $N \supset [\pi_2, \pi_2] = \langle t_1^2,\ t_2^2,\ t_3^{2n} \rangle$. Thus we obtain the following result, which is the same as Theorem 3.3 of [1].

Corollary 2.5. *The following table gives a complete list of all free actions(up to topological conjugacy) of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi$.*

groups G	AC classes of normal nilpoent subgroups	
$\mathbb{Z}_{\frac{4n}{p}}$	$\frac{2n}{p} \in \mathbb{N}$	$N_1 = \langle t_1,\ t_2,\ t_3^{\frac{2n}{p}} \rangle$
	$\frac{n}{p} \in \mathbb{N},\ p \ne 1$	$N_2 = \langle t_1,\ t_2 t_3^{\frac{n}{p}},\ t_3^{\frac{2n}{p}} \rangle$
		$N_3 = \langle t_1 t_3^{\frac{n}{p}},\ t_2 t_3^{\frac{n}{p}},\ t_3^{\frac{2n}{p}} \rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_{\frac{8n}{p}}$	$\frac{4n}{p} \in \mathbb{N},\ p \in 2\mathbb{N}$	$L_1 = \langle t_1^2,\ t_2,\ t_3^{\frac{4n}{p}} \rangle$
	$\frac{2n}{p} \in \mathbb{N},\ p \in 2\mathbb{N}+2$	$L_2 = \langle t_1^2,\ t_2 t_3^{\frac{2n}{p}},\ t_3^{\frac{4n}{p}} \rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{\frac{16n}{p}}$	$\frac{8n}{p} \in \mathbb{N},\ p \in 4\mathbb{N}$	$N = \langle t_1^2,\ t_2^2,\ t_3^{\frac{8n}{p}} \rangle$

Proof. Let N be a normal nilpotent subgroup of π isomorphic to Γ_p with π/N abelian. Then by Lemma 2.2, we have

$$N = \langle\, t_1^{d_1} t_2^{m} t_3^{\ell},\, t_2^{d_2} t_3^{r},\, t_3^{\frac{2nd_1d_2}{p}} \,\rangle,$$

where $d_1,\, d_2$ are divisors of p; $0 \le m < \bar{d} = (d_1, d_2)$, $\frac{pm}{d_1 d_2} \in \mathbb{Z}$.

Since $[\pi_2, \pi_2] = \langle t_1^2,\ t_2^2,\ t_3^{2n}\rangle \subset N$, we have $(d_1,\, d_2)$ are $(1,1)$, $(2,1)$, $(1,2)$ and $(2,2)$. We shall deal only with the case $d_1 = d_2 = 1$, because the other cases can be done similarly.

Since $\bar{d} = (d_1, d_2) = 1$, we get $m = 0$ by Lemma 2.2. Therefore the possible normal nilpotent subgroups are

$$N_1 = \langle t_1,\ t_2,\ t_3^{\frac{2n}{p}}\rangle, \qquad N_2 = \langle t_1,\ t_2 t_3^{\frac{n}{p}},\ t_3^{\frac{2n}{p}}\rangle,$$
$$N_3 = \langle t_1 t_3^{\frac{n}{p}},\ t_2 t_3^{\frac{n}{p}},\ t_3^{\frac{2n}{p}}\rangle, \qquad N_4 = \langle t_1 t_3^{\frac{n}{p}},\ t_2,\ t_3^{\frac{2n}{p}}\rangle.$$

When $p = 1$, we have by Theorem 2.4,

$$N_1 \sim N_2 \sim N_3 \sim N_4,$$

which is the same result in [2]. When $p > 1$, by applying Theorem 2.4, we can conclude that

(1) $N_1 \nsim N_2$,

(3) $N_1 \nsim N_4$,

(4) $N_2 \sim N_3$

(6) $N_3 \nsim N_4$

Therefore there exist 3 affinely non-conjugate normal nilpotent subgroups N_i $(i = 1, 2, 3)$ of π. □

References

1. D. Choi and J.K. Shin, *Free actions of finite abelian groups on 3-dimensional nilmanifolds*, J. Korean Math. Soc. **42** (2005), 795–826.
2. H.Y. Chu and J.K. Shin, *Free actions of finite groups on the 3-dimensional nilmanifold*, Topology Appl. **144** (2004), 255–270.

3. K. Dekimpe, P. Igodt, S. Kim and K.B. Lee, *Affine structures for closed 3-dimensional manifolds with nil-geometry*, Quarterly J. Math. **46** (1995), 141–167.
4. K.Y. Ha, J.H. Jo, S.W. Kim and J.B. Lee, *Classification of free actions of finite groups on the 3-torus*, Topology Appl. **121** (2002), 469–507.
5. W. Heil, *On P^2-irreducible 3-manifolds*, Bull. Amer. Math. Soc. **75** (1969), 772–775.
6. W. Heil, *Almost sufficiently large Seifert fiber spaces*, Michigan Math. J. **20** (1973), 217–223.
7. J. Hempel, *Free cyclic actions of $S^1 \times S^1 \times S^1$*, Proc. Amer. Math. Soc. **48** (1975), 221–227.
8. K.B. Lee, J.K. Shin and Y. Shoji, *Free actions of finite abelian groups on the 3-torus*, Topology Appl. **53** (1993), 153–175.
9. J.K. Shin, *Isometry groups of unimodular simply connected 3-dimensional Lie groups*, Geom. Dedicata **65** (1997), 267–290.
10. F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. Math. **87** (1968), 56–88.
11. S. Wolfram, *Mathematica*, Wolfram Research, 1993.

*
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea

**
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea

E-mail: jkshin@cnu.ac.kr