TRANSITIVE SETS WITH C^1 -STABLY LIMIT SHADOWING

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ABSTRACT. We show that a nontrivial transitive set is C^1 -stably limit shadowable if and only if the transitive set is hyperbolic.

1. Introduction

Let M be a closed C^{∞} manifold, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1 -topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in Diff(M)$. In the dynamical systems, the shadowing property is very useful notion. Actually, it deals with the stability theorem (see [4]). For instance, Sakai [6] showed that f belongs to the C^1 -interior of the set of a diffeomorphism having the shadowing property if and only if f is structurally stable. In this paper, we deal with the another shadowing property, that is, the limit shadowing property which was introduced by Lee [1]. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b(-\infty \leq a < b \leq \infty)$ in M is called a δ -pseudo orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. Let $\Lambda \subset M$ be a closed f-invariant set. We say that f has the limit shadowing property on Λ (or Λ is limit shadowable for f) if there exists a $\delta > 0$ with the following property: if a sequence $\{x_i\}_{i\in\mathbb{Z}}\subset\Lambda$ is a δ -pseudo orbit of f for which relations $d(f(x_i), x_{i+1}) \to 0$ as $i \to +\infty$, and $d(f^{-1}(x_{i+1}), x_i) \to 0$ as

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 $i \to -\infty$ hold, then there is a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm \infty$. We say that Λ is locally maximal if there is a neighborhood U of Λ such that $\bigcap_{n \in \mathbb{Z}} f^n(U) = \Lambda$. We say that Λ is transitive if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. We say that Λ is nontrivial if Λ is not just a periodic orbit. We say that f has the C^1 -stably limit shadowing property on Λ (or Λ is the C^1 -stably limit shadowable for f) if there are a C^1 -neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that

- (1) $\Lambda = \Lambda_f(U) = \bigcap_{n \in \mathbb{Z}} f^n(U)$ (locally maximal),
- (2) for any $g \in \mathcal{U}(f)$, g has the limit shadowing property on $\Lambda_g(U)$, where $\Lambda_g(U) = \bigcap_{i \in \mathbb{Z}} g^n(U)$ is the continuation of $\Lambda = \Lambda_f(U)$.

We say that Λ is hyperbolic if the tangent bundle $T_{\Lambda}M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exists constants C > 0 and $0 < \lambda < 1$ such that

$$||D_x f^n|_{E_x^s}|| \le C\lambda^n$$
 and $||D_x f^{-n}|_{E_x^u}|| \le C\lambda^n$

for all $x \in \Lambda$ and $n \geq 0$. Moreover, we say that Λ admits a dominated splitting if the tangent bundle $T_{\Lambda}M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants C > 0 and $0 < \lambda < 1$ such that

$$||D_x f^n|_{E(x)}|| \cdot ||D_x f^{-n}|_{F(f^n(x))}|| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. In this paper, we study the relations between the C^1 -stably limit shadowing property and hyperbolic. For the above definition, in [2] the authors showed that f has the C^1 -stably shadowing property on the chain component $C_f(p)$ containing hyperbolic saddle p if and only if $C_f(p)$ is hyperbolic. In this paper, we use the above definition on the nontrivial transitive set. The following is main theorem in this paper.

THEOREM 1.1. Let Λ be a nontrivial transitive set of $f \in \text{Diff}(M)$. f has the C^1 -stably limit shadowing property on Λ if and only if it is hyperbolic.

In [1], Lee showed that if Λ is hyperbolic then it is limit shadowable. And by the hyperbolicity, f has the C^1 -stably limit shadowing property. Thus in this paper we show that if f has the C^1 -stably limit shadowing property on transitive sets, then it is hyperbolic.

2. Proof of Theorem 1.1

Let M be as before, and let $f \in \text{Diff}(M)$. The following lemma is obtained by Pugh's closing lemma.

LEMMA 2.1. [5] Let Λ be a nontrivial transitive set. There exist a sequence of diffeomorphisms $\{g_n\}_{n\in\mathbb{N}}$ and periodic orbit P_n of g_n with period $\pi(P_n)$ as $n\to\infty$ such that $g_n\to f$ with C^1 -topology and $\lim P_n=\Lambda$

From the above lemma, if Λ is a locally maximal transitive set, then we can take a periodic point $p \in P(f)$ such that $\mathcal{O}_f(p) \subset U$, where U is a compact neighborhood of Λ . Since Λ is locally maximal in U, we know that $p \in \Lambda$.

Remark 2.2. We know that

- (a) Let I be the unit interval. If $f: I \to I$ is an identity map, then f does not have the limit shadowing property.
- (b) Let S^1 be the unit circle. If $f: S^1 \to S^1$ is an irrational rotation then f does not have the limit shadowing property.

The following so-called Franks' lemma will play essential roles in our proof.

LEMMA 2.3. Let $\mathcal{U}(f)$ be any given C^1 -neighborhood of f. Then there exist $\epsilon > 0$ and a C^1 -neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_N\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $g' \in \mathcal{U}(f)$ such that g'(x) = g(x) if $x \in \{x_1, x_2, \ldots, x_N\} \cup (M \setminus U)$ and $D_{x_i}g' = L_i$ for all $1 \le i \le N$.

LEMMA 2.4. Let $f \in \text{Diff}(M)$, and let Λ be a compact f-invariant set. Suppose that f has the C^1 -stably limit shadowing property on Λ . Then there exists a C^1 -neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}_0(f)$, every $p \in \Lambda_g(U) \cap P(g)$ is hyperbolic for g, where $\Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U)$.

Proof. Since f has the C^1 -stably limit shadowing property on Λ , there exist a compact neighborhood U of Λ and a C^1 -neighborhood U(f) of f such that for any $g \in U(f)$, g has the limit shadowing property on $\Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U)$. Let $\epsilon > 0$ and $U_0(f) \subset U(f)$ be the corresponding number and C^1 -neighborhood of f given by Lemma 2.3 with respect to U(f). Then we obtained a C^1 neighborhood $U_0(f)$. Suppose that there

exists a non-hyperbolic periodic point $p \in \Lambda_g(U)$ for some $g \in \mathcal{U}_0(f)$. Note that since Λ is locally maximal (reducing $\mathcal{U}_0(f)$ if necessary), we may assume that q is contained in the interior of U. To simplify the notions, we may assume that g(p) = p. Then by making use of Lemma 2.3, we take a linear isomorphism $L: T_pM \to T_pM$ such that $||L - D_pg|| < \epsilon$. We can choose $\alpha > 0$ with $B_{4\alpha}(p) \subset U$ and $g_1 C^1$ -nearby g such that

$$g_1(x) = \begin{cases} \exp_p \circ L \circ \exp_p^{-1} & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

Clearly, $g_1(p) = g(p)$. Then if λ is real then there is a g_1 -invariant normally hyperbolic small arc \mathcal{I}_p center at p such that $g_1^k|_{\mathcal{I}_p} = id$ for some k > 0. And if λ is complex then a g_1 -invariant normally hyperbolic small circle S_p with a small diameter center at p such that $g_1|_{S}$ is conjugated to an irrational rotation map. Since \mathcal{I}_p and \mathcal{S}_p are g_1 -invariant, we see that $\mathcal{I}_p \subset \Lambda_{g_1}(U)$ and $\mathcal{S}_p \subset \Lambda_{g_1}(U)$. Since g_1 has the limit shadowing property on $\Lambda_{q_1}(U)$, both $g_1^k|_{\mathcal{I}_p}$ and $g_1|_{\mathcal{S}_p}$ must have the limit shadowing property. Since \mathcal{I}_p and \mathcal{S}_p are g_1 -invariant normally hyperbolic, the shadowing point belongs to \mathcal{I}_p and \mathcal{S}_p . If not, then we show that a contradiction. Let $y \in M$ be a shadowing point. Since $y \notin \mathcal{I}_p$, by hyperbolicity there is $l \in \mathbb{Z}$ such that for any $\eta > 0$, $f^l(y) \notin B_{\eta}(\mathcal{I}_p)$. Moreover, if l > 0 then for all $i \geq 0$, $f^{l+i}(y) \notin B_{\eta}(\mathcal{I}_p)$. Therefore, $d(f^i(y), x_i) \neq 0$ as $i \to +\infty$. This is a contradiction. Thus we know that the shadowing point belongs to \mathcal{I}_p . Similarly we get the same result for S_p . By Remark 2.2, this is a contradiction. This completes the proof of Lemma 2.4.

From Lemma 2.4, the family of periodic sequences of linear isomorphisms of \mathbb{R}^{dimM} generated by $Dg(g \in \mathcal{U}_0(f))$ along the hyperbolic periodic points $p \in \Lambda \cap P(g)$ is uniformly hyperbolic. That is, there exists $\epsilon > 0$ such that for any $g \in \mathcal{U}_0(f), p \in \Lambda \cap P(g)$, and any sequence of linear maps $L_i : T_{g^i(p)}M \to T_{g^{i+1}(p)}M$ with $||L_i - D_{g^i(p)}g|| < \epsilon$ for $1 \le i \le \pi(p) - 1$, and $\prod_{i=0}^{\pi(p)-1} L_i$ is hyperbolic. Here $\mathcal{U}_0(f)$ is the C^1 -neighborhood of f given by Lemma 2.4 with respect to $\mathcal{U}(f)$. Thus by Proposition II.1 in [3] and the above Lemma 2.4, we get the following.

PROPOSITION 2.5. Suppose that f has the C^1 -stably limit shadowing property on Λ and let $\mathcal{U}_0(f)$ as in the Lemma 2.4. Then there are constants C > 0, $\lambda \in (0,1)$ and m > 0 such that

(a) for any $g \in \mathcal{U}_0(f)$, if $p \in \Lambda \cap P(g)$ has a minimum period $\pi(p) \geq m$, then

$$\prod_{i=0}^{k-1} \|D_{g^{im}(p)}g^m|_{E^s_{g^{im}(p)}}\| < C\lambda^k \text{ and } \prod_{i=0}^{k-1} \|D_{g^{-im}(p)}g^{-m}|_{E^u_{g^{-im}(p)}}\| < C\lambda^k,$$

where $k = [\pi(p)/m]$.

(b) Λ admits a dominated splitting $T_{\Lambda}M = E \oplus F$ with $\dim E = \operatorname{index}(p)$.

It is well known that if p is a hyperbolic periodic point of f with period k then the sets

$$W^{s}(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \text{ and}$$

$$W^{u}(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1 -injectively immersed submanifolds of M. Let q be a hyperbolic periodic point of f. We say that p and q are homoclinically related, and write $p \sim q$ if

$$W^{s}(p) \cap W^{u}(q) \neq \emptyset$$
 and $W^{u}(p) \cap W^{s}(q) \neq \emptyset$.

It is clear that if $p \sim q$ then $\mathrm{index}(p) = \mathrm{index}(q)$; that is, $\mathrm{dim}W^s(p) = \mathrm{dim}W^s(q)$. Let p be a hyperbolic periodic point of f, and let Λ be a transitive set.

PROPOSITION 2.6. Suppose that f has the C^1 -stably limit shadowing property on Λ . Then for any hyperbolic point $q \in \Lambda \cap P(f)$,

$$index(p) = index(q).$$

To prove Proposition 2.6, we need the following lemma.

LEMMA 2.7. Let Λ be a transitive set of f. Suppose that f has the limit shadowing property on Λ . Then for any $p, q \in \Lambda \cap P_h(f)$,

$$W^s(p) \cap W^u(q) \neq \emptyset$$
 and $W^u(p) \cap W^s(q) \neq \emptyset$,

where $P_h(f)$ is the set of hyperbolic periodic points of f.

Proof. In this proof, we will show that $W^u(p) \cap W^s(q) \neq \emptyset$. The other case is similar. Let p,q be two hyperbolic periodic points of f in Λ and let $\epsilon(p) > 0$ and $\epsilon(q) > 0$ be as before with respect to p and q. Fix $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$. To simplify the notions in the proof, we assume that f(p) = p and f(q) = q. Let $0 < \delta < \epsilon/2$ be the number of the limit shadowing property of $f|_{\Lambda}$. Since Λ is a transitive set, there is $x \in \Lambda$ such that $\omega(x) = \Lambda$. For the above $\delta > 0$, we can choose $l_1 > 0$ and $l_2 > 0$ such that $d(f^{l_1}(x), p) < \delta/2$ and $d(f^{l_2}(x), q) < \delta/2$. We may assume

that $l_2 > l_1 > 0$. Then we can construct a δ -limit pseudo orbit of f as follows: (i) $f^i(p) = x_i$ for $i \geq 0$, (ii) $f^{l_1+i}(x) = x_i$ for $0 \leq i \leq l_2 - 1$, and (iii) $f^i(q) = x_i$ for $l_2 \leq i$. Then we obtained a δ -limit pseudo orbit of f,

$$\xi = \{\ldots, p, p, x_1, \ldots, x_{l_2-1}, q, q, \ldots\}.$$

Since f has the limit shadowing property on Λ , we can choose a point $y \in M$ such that $d(f^n(y), x_n) \to 0$ as $n \to \pm \infty$. Then we choose $n_1 > 0$ sufficiently large such that $f^{-n}(y) \in W^u_{\epsilon}(p)$ and $f^n(y) \in W^s_{\epsilon}(q)$, for all $n \ge n_1$. Therefore, $y \in f^n(W^u_{\epsilon}(p))$ and $y \in f^{-n}(W^s_{\epsilon}(q))$. Thus $y \in W^u(p) \cap W^s(q)$. Consequently, one can get $W^u(p) \cap W^s(q) \neq \emptyset$. \square

If $p \in P(f)$ is hyperbolic, then for any $g \in \text{Diff}(M)$ C^1 -nearby f, there exists a unique hyperbolic periodic point $p_g \in P(g)$ nearby p such that $\pi(p_g) = \pi(p)$ and $\operatorname{index}(p_g) = \operatorname{index}(p)$. Such a p_g is called the *continuation of* p. It is well known that a dominated splitting always extends to a neighborhood. More precisely, let Λ be a closed $f \in \operatorname{Diff}(M)$ -invariant set. Then if Λ admits a dominated splitting $T_{\Lambda}M = E \oplus F$ such that $\dim E_x(x \in \Lambda)$ is constant, then there are a C^1 -neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood f of f such that for any f is f admits a dominated splitting

$$T_{\bigcap_{n\in\mathbb{Z}}g^n(U)}M=E'(g)\oplus F'(g)$$

with $\dim E'(g) = \dim E$.

We say that f is Kupka-Smale if every periodic point is hyperbolic and for any $p, q \in P(f)$, $W^s(p)$ and $W^u(q)$ are transverse and $W^u(p)$ and $W^s(q)$ intersect transversally. Note that the Kupka-Smale diffeomorphism is a residual subset of Diff(M).

Proof of Proposition 2.6. Suppose f has the C^1 -stably limit shadowing property on Λ . Then there is a C^1 -neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ has the limit shadowing property. Assume that Proposition 2.6 is not true. Then there is $p, q \in \Lambda \cap P(f)$ such that $\operatorname{index}(p) \neq \operatorname{index}(q)$. Thus we know that $\dim W^u(q) + \dim W^s(p) < \dim M$ or $\dim W^s(q) + \dim W^u(p) < \dim M$. Without loss of generality, we may assume that $\dim W^s(p) + \dim W^u(q) < \dim M$. Since f has the C^1 -stably limit shadowing property on Λ , we take a Kupka-Smale diffeomorphism $g \in \mathcal{U}(f)$. Then g has the limit shadowing property on $\Lambda_g(U)$ and $p_g, q_g \in \Lambda_g(U)$, where p_g and q_g are the continuation of p and q, respectively. One can see that $\dim W^s(p_g) = \dim W^s(p)$ and $\dim W^u(q_g) = \dim W^u(q)$. Since g is Kupka-Smale, $W^s(p_g) \cap W^u(q_g) = \emptyset$.

This is a contradiction by Lemma 2.7.

Let us recall Mañé's ergodic closing lemma in [3]. For any $\epsilon > 0$, let $B_{\epsilon}(f,x)$ be an ϵ -tubular neighborhood of f-orbit of x, that is, $B_{\epsilon}(f,x) = \{y \in M : d(f^{n}(x), y) < \epsilon \text{ for some } n \in \mathbb{Z}\}$. Let Σ_{f} be the set of points $x \in M$ such that for any C^{1} -neighborhood $\mathcal{U}(f)$ of f and $\epsilon > 0$, there are $g \in \mathcal{U}(f)$ and $y \in P(g)$ satisfying g = f on $M \setminus B_{\epsilon}(f,x)$ and $d(f^{i}(x), g^{i}(y)) \leq \epsilon$ for $0 \leq i \leq \pi(y)$.

REMARK 2.8. ([3, Theorem A]). For any f-invariant probability measure μ , we have $\mu(\Sigma_f) = 1$.

By Lemma 2.4 and Pugh's closing Lemma, we see that $\overline{P(f) \cap \Lambda} = \Lambda$.

Proof of Theorem 1.1. Let $\mathcal{U}_0(f)$ be the C^1 -neighborhood of f given by Proposition 2.5. To get the conclusion, it is sufficient to show that $\Lambda_i(f)$ is hyperbolic, where $\Lambda_i(f) = \overline{P_i(f|_{\Lambda})}$, and i is an index of Λ . Fix any neighborhood $U_i \subset U$ of $\Lambda_i(f)$. Note that by Proposition 2.6, $\Lambda_j(f) = \overline{P_j(f|_{\Lambda})} = \emptyset$ if $i \neq j$. Thus we show that the following: let $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ be a small connected C^1 -neighborhood of f. If any $g \in \mathcal{V}(f)$ satisfies q = f on $M \setminus U_i$, then index (p) = index (q) for any $p, q \in \Lambda_g(U) \cap P(g)$. Indeed, suppose not, then there are $g_1 \in \mathcal{V}(f)$ and $q \in \Lambda_g(U) \cap P(g_1)$ such that $g_1 = f$ on $M \setminus U_i$ and index $(p) \neq \text{index } (q)$. Suppose that $g_1^n(q) = q, k = \text{index}(q)$, and define $\gamma : \mathcal{V}(f) \to \mathbb{Z}$ by

$$\gamma(g) = \sharp \{ y \in \Lambda_q(U) \cap P(g) : g^n(y) = y \text{ and } \operatorname{index}(y) = k \}.$$

By Lemma 2.4, the function γ is continuous, and since $\mathcal{V}(f)$ is connected, it is constant. But the property of g_1 implies $\gamma(g_1) > \gamma(f)$. This is a contradiction. We will finish the proof of Theorem 1.1. Then we use the proof of Theorem B in [3]. Thus we show that

$$\liminf_{n \to \infty} ||D_x f^n|_{E_x}|| = 0 \text{ and } \liminf_{n \to \infty} ||D_x f^{-n}|_{F_x}|| = 0$$

for all $x \in \Lambda$, and thus the splitting is hyperbolic. More precisely, we will prove the case of $\liminf_{n\to\infty} \|D_x f_{|E}^n\| = 0$ (the other case is similar). We will derive a contraction. If it is not true, then there is $x \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|Df^m|_{E_{f^{mj}(x)}}\| \ge 1$$

for all $n \geq 0$. Thus

$$\frac{1}{n} \sum_{i=0}^{n-1} \log ||Df^m||_{E_{f^{mj}(x)}}|| \ge 0$$

for all $n \geq 0$. The proof is similar to end of the proof of Theorem 1.3 (see [2]). Then we know that $\liminf_{n\to\infty} \|D_x f^n|_{E_x}\| = 0$ for all $x \in \Lambda$. Thus Λ is hyperbolic. This completes the proof of the "only if" part of Theorem 1.1.

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