

FIXED POINT AND PERIODIC POINT THEOREMS ON METRIC SPACES

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ABSTRACT. The aim of this paper is to establish a new fixed point theorem for a set-valued mapping defined on a metric space satisfying a weak contractive type condition and to establish a new common fixed point theorem for a pair of set-valued mappings defined on a metric space satisfying a weak contractive type inequality. And we give periodic point theorems for single-valued mappings defined on a metric space satisfying weak contractive type conditions.

1. Introduction

Banach's contraction principle [5] is one of the pivotal results of analysis. It is widely recognized as the source of metric fixed point theory. Banach's contraction principle including its several generalizations for single-valued and set-valued mappings in metric spaces plays an important role in several branches of mathematics. For instance, it has been used to research many problems in nonlinear analysis and to study the convergence of algorithms in computational mathematics. Also, Banach's contraction principle is a powerful tool in the study on finding fixed points of mappings defined on metric spaces. Banach's contraction principle has been generalized and extended in many directions [2, 3, 4, 6, 9, 13, 14, 15, 17, 18, 19, 20, 21, 23].

The authors [1] introduced weak contraction principle in Hilbert spaces, which is a generalization of Banach's contraction principle. And then, the author [22] extended this principle to metric spaces. The authors [10, 7, 11, 12, 24] obtained fixed point results involving weak contractions and mappings satisfying weak contractive type inequalities.

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The author [18] used a control function which alters the distance between two points in a metric space to obtained fixed point results. Such a control function called an altering distance function.

Recently, the authors [8] introduced the notion of a generalized weakly contractive mapping by using an altering distance function, and gave some fixed point theorems of this mapping.

In this paper we prove a new fixed point theorem for a set-valued mapping defined on a metric space satisfying a weak contractive type inequality, and prove a new common fixed point theorem for a pair of set-valued mappings defined on a metric space satisfying a weak contractive type inequality. And then we obtain extensions of Theorem 3.1 and Theorem 3.2 in [8] to the case of set-valued mappings, and we have generalizations of Theorem 3.1 and Theorem 3.2 in [8]. Also, we give periodic point theorems for single-valued mappings defined on a metric space satisfying weak contractive type inequalities.

We recall some definitions and notations in the following.

Let (X, d) be a metric space. We denote by $K(X)$ the family of non-empty compact subsets of (X, d) . Let $H(\cdot, \cdot)$ be the Hausdorff distance on $K(X)$, i.e.,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}, \text{ for } A, B \in K(X),$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the subset B .

For $A, B \in K(X)$, let $D(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$.

Then we have $D(A, B) \leq H(A, B)$ for all $A, B \in K(X)$.

Let X be a non-empty set, and let $S, T : X \rightarrow 2^X$ be set-valued mappings. Then $z \in X$ is called a *fixed point* of T if $z \in Tz$, and $z \in X$ is called a *common fixed point* of S and T if $z \in Sz \cap Tz$.

A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is an *altering distance function* [18] if the following conditions are satisfied:

- (i) ψ is monotone increasing and continuous;
- (ii) $\psi(t) = 0$ if and only if $t = 0$.

From now on, let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuous function such that $\phi(t) = 0$ if and only if $t = 0$.

2. Fixed point and common fixed point theorems

THEOREM 2.1. *Let (X, d) be a complete metric space. Suppose that a set-valued mapping $T : X \rightarrow K(X)$ satisfies*

$$\begin{aligned} \psi(H(Tx, Ty)) &\leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), \\ &\quad \frac{1}{2}\{d(x, Ty) + d(y, Tx)\}\}) - \phi(\max\{d(x, y), d(y, Ty)\}), \end{aligned} \quad (2.1)$$

for all $x, y \in X$, where ψ is an altering distance function.

Then T has a fixed point in X .

Proof. Let $x_0 \in X$ be fixed. Then we can find $x_1 \in Tx_0$ such that $d(x_0, x_1) = d(x_0, Tx_0)$, because $Tx_0 \in K(X)$. For x_1 also, we can find $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$.

Continuing this process, we can find a sequence $\{x_n\}$ of points in X such that

$$x_{n+1} \in Tx_n \text{ and } d(x_n, x_{n+1}) = d(x_n, Tx_n) \text{ for all } n \geq 0.$$

If there exists a positive integer N such that $x_N = x_{N+1}$, then $d(x_N, Tx_N) \leq d(x_N, x_{N+1}) = 0$. Hence x_N is a fixed point of T .

Thus we may assume that $x_n \neq x_{n+1}$ for all $n \geq 0$.

For $x = x_{n-1}$ and $y = x_n$ in (2.1), we obtain

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(x_n, Tx_n)) \\ &\leq \psi(H(Tx_{n-1}, Tx_n)) \leq \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\quad \frac{1}{2}\{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})\}\}) - \phi(\max\{d(x_{n-1}, x_n), d(x_n, Tx_n)\}) \\ &\leq \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad \frac{1}{2}\{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\}\}) - \phi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \\ &\leq \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad \frac{1}{2}\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}\}) - \phi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned}$$

Suppose that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$. Then we have $\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1}))$. Hence $\phi(d(x_n, x_{n+1})) \leq 0$. Thus, $d(x_n, x_{n+1}) = 0$ or $x_n = x_{n+1}$, which is a contradiction.

Therefore, we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \text{ for all } n \geq 0 \quad (2.2)$$

and so

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)). \quad (2.3)$$

From (2.2) the sequence $\{d(x_n, x_{n+1})\}$ is a non-decreasing sequence of non-negative real numbers, and hence there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r.$$

Letting $n \rightarrow \infty$ in (2.3) and using continuity of ψ and ϕ , we obtain $\psi(r) \leq \psi(r) - \phi(r)$. Hence $\phi(r) \leq 0$, and hence $r = 0$. Thus we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (2.4)$$

We now show that $\{x_n\}$ is a Cauchy sequence in X .

Assume that $\{x_n\}$ is not a Cauchy sequence.

Then there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$, there exists $m(k) > n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } d(x_{m(k)-1}, x_{n(k)}) < \epsilon.$$

Thus we have

$$\begin{aligned} \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &< d(x_{m(k)}, x_{m(k)-1}) + \epsilon. \end{aligned}$$

Letting $k \rightarrow \infty$ in above inequality, we obtain

$$\lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \quad (2.5)$$

Since $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$, we have

$$\lim_{k \rightarrow \infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon, \quad (2.6)$$

and

$$\lim_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)-1}) = \epsilon. \quad (2.7)$$

And also, we have

$$\begin{aligned} &d(x_{n(k)}, x_{m(k)-1}) \\ &\leq d(x_{n(k)}, Tx_{n(k)}) + d(Tx_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}) \\ &\leq d(x_{n(k)}, x_{n(k)+1}) + d(Tx_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}) \\ &\text{and} \\ &d(x_{m(k)}, Tx_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)}, Tx_{n(k)}) \\ &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}). \end{aligned}$$

Taking limit as $k \rightarrow \infty$ in above inequalities and using (2.4), (2.5) and (2.7), we obtain

$$\lim_{k \rightarrow \infty} d(x_{m(k)}, Tx_{n(k)}) = \epsilon. \quad (2.8)$$

For $x = x_{m(k)-1}$ and $y = x_{n(k)}$ in (2.1), we have

$$\begin{aligned} & \psi(d(x_{m(k)}, Tx_{n(k)})) \\ & \leq \psi(H(Tx_{m(k)-1}, Tx_{n(k)})) \\ & \leq \psi(\max\{d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)-1}, Tx_{m(k)-1}), d(x_{n(k)}, Tx_{n(k)}), \\ & \quad \frac{1}{2}\{d(x_{m(k)-1}, Tx_{n(k)}) + d(x_{n(k)}, Tx_{m(k)-1})\}\}) \\ & \quad - \phi(\max\{d(x_{m(k)-1}, x_{n(k)}), d(x_{n(k)}, Tx_{n(k)})\}) \\ & \leq \psi(\max\{d(x_{m(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), \\ & \quad \frac{1}{2}\{d(x_{m(k)-1}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)})\}\}) \\ & \quad - \phi(\max\{d(x_{m(k)-1}, x_{n(k)}), d(x_{n(k)}, x_{n(k)+1})\}). \end{aligned}$$

Letting $k \rightarrow \infty$ in above inequality and using (2.4), (2.5), (2.6), (2.7) and (2.8), we obtain $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$. Hence $\phi(\epsilon) \leq 0$, and hence $\epsilon = 0$, which is a contradiction.

Therefore, $\{x_n\}$ is a Cauchy sequence in X .

Since X is complete, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$.

For $x = x_n$ and $y = z$ in (2.1), we obtain

$$\begin{aligned} & \psi(d(x_{n+1}, Tz)) \\ & \leq \psi(H(Tx_n, Tz)) \leq \psi(\max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \\ & \quad \frac{1}{2}\{d(x_n, Tz) + d(z, Tx_n)\}\}) - \phi(\max\{d(x_n, z), d(z, Tz)\}) \\ & \leq \psi(\max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), \\ & \quad \frac{1}{2}\{d(x_n, Tz) + d(z, x_{n+1})\}\}) - \phi(\max\{d(x_n, z), d(z, Tz)\}). \end{aligned}$$

Letting $n \rightarrow \infty$ in above inequality, we have $\psi(d(z, Tz)) \leq \psi(d(z, Tz)) - \phi(d(z, Tz))$, which implies that $\phi(d(z, Tz)) \leq 0$. Hence $d(z, Tz) = 0$, and hence $z \in Tz$. \square

REMARK 2.2. The above result is an extension of Theorem 3.1 in [8] to the case of set-valued mapping.

From Theorem 2.1 we have the following two corollaries.

COROLLARY 2.3. Let (X, d) be a complete metric space. Suppose that a set-valued mapping $T : X \rightarrow K(X)$ satisfies

$$\begin{aligned} \psi(H(Tx, Ty)) &\leq \psi(\max\{d(x, y), \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \\ &\quad \frac{1}{2}\{d(x, Ty) + d(y, Tx)\}\}) - \phi(\max\{d(x, y), d(y, Ty)\}), \end{aligned}$$

for all $x, y \in X$, where ψ is an altering distance function.

Then T has a fixed point in X .

COROLLARY 2.4. Let (X, d) be a complete metric space. Suppose that a set-valued mapping $T : X \rightarrow K(X)$ satisfies

$$\begin{aligned} \psi(H(Tx, Ty)) &\leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty)\}) \\ &\quad - \phi(\max\{d(x, y), d(y, Ty)\}), \end{aligned}$$

for all $x, y \in X$, where ψ is an altering distance function.

Then T has a fixed point in X .

In Theorem 2.1, if T is a single valued mapping, then we have the following corollary.

COROLLARY 2.5. [8] Let (X, d) be a complete metric space. Suppose that a mapping $f : X \rightarrow X$ satisfies

$$\begin{aligned} \psi(d(fx, fy)) &\leq \psi(\max\{d(x, y), d(x, fx), d(y, fy), \\ &\quad \frac{1}{2}\{d(x, fy) + d(y, fx)\}\}) - \phi(\max\{d(x, y), d(y, fy)\}), \end{aligned} \quad (2.9)$$

for all $x, y \in X$, where ψ is an altering distance function.

Then f has a unique fixed point in X .

Proof. From Theorem 2.1 f has a fixed point in X . It follows from (2.9) that the fixed point of f is unique. \square

COROLLARY 2.6. Let (X, d) be a complete metric space. Suppose that a mapping $f : X \rightarrow X$ satisfies

$$\begin{aligned} \psi(d(fx, fy)) &\leq \psi(\max\{d(x, y), \frac{1}{2}\{d(x, fx) + d(y, fy)\}, \\ &\quad \frac{1}{2}\{d(x, fy) + d(y, fx)\}\}) - \phi(\max\{d(x, y), d(y, fy)\}), \end{aligned} \quad (2.10)$$

for all $x, y \in X$, where ψ is an altering distance function.

Then f has a unique fixed point in X .

From Corollary 2.5 we have the following result.

COROLLARY 2.7. *Let (X, d) be a complete metric space. Suppose that a mapping $f : X \rightarrow X$ satisfies*

$$\begin{aligned} & \psi(d(fx, fy)) \\ & \leq \psi(\max\{d(x, y), d(x, fx), d(y, fy)\}) - \phi(\max\{d(x, y), d(y, fy)\}), \end{aligned} \quad (2.11)$$

for all $x, y \in X$, where ψ is an altering distance function.

Then f has a unique fixed point in X .

THEOREM 2.8. *Let (X, d) be a complete metric space. Suppose that set-valued mappings $S, T : X \rightarrow K(X)$ satisfy*

$$\begin{aligned} & \psi(H(Sx, Ty)) \leq \psi(\max\{d(x, y), d(x, Sx), d(y, Ty), \\ & \frac{1}{2}\{d(x, Ty) + d(y, Sx)\}\}) - \phi(\max\{d(x, y), d(x, Sx), d(y, Ty)\}), \end{aligned} \quad (2.12)$$

for all $x, y \in X$, where ψ is an altering distance function.

Then S and T have a fixed point in X . Moreover, any fixed point of S is a fixed point of T and conversely.

Proof. Suppose that p is a fixed point of S .

Then from (2.12) we have

$$\begin{aligned} & \psi(d(p, Tp)) \leq \psi(H(Sp, Tp)) \\ & \leq \psi(\max\{d(p, p), d(p, Sp), d(p, Tp), \frac{1}{2}\{d(p, Tp) + d(p, Sp)\}\}) \\ & \quad - \phi(\max\{d(p, p), d(p, Sp), d(p, Tp)\}) \\ & = \psi(\max\{0, 0, d(p, Tp), \frac{1}{2}d(p, Tp)\}) - \phi(0, 0, d(p, Tp)) \\ & = \psi(d(p, Tp)) - \phi(d(p, Tp)) \end{aligned}$$

which implies $\phi(d(p, Tp)) = 0$, and so $d(p, Tp) = 0$. Thus, $p \in Tp$ and p is a fixed point of T .

Using a similar argument, we have that any fixed point of T is a fixed point of S .

Let $x_0 \in X$ be fixed. Then we can find $x_1 \in Sx_0$ such that $d(x_0, x_1) = d(x_0, Sx_0)$. Again, we can find $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$.

Continuing this process, we can find a sequence $\{x_n\}$ of points in X such that

$$\begin{aligned} & x_{2n+1} \in Sx_{2n}, \quad x_{2n+2} \in Tx_{2n+1}, \quad d(x_{2n}, x_{2n+1}) = d(x_{2n}, Sx_{2n}) \\ & \text{and } d(x_{2n+1}, x_{2n+2}) = d(x_{2n+1}, Tx_{2n+1}) \text{ for all } n \geq 0. \end{aligned}$$

If there exists a positive integer N such that $x_{2N} = x_{2N+1}$, then $x_{2N} \in Sx_{2N}$. Thus $x_{2N} \in Tx_{2N}$. Hence x_{2N} is a common fixed point of S and T .

Therefore, we may assume that $x_n \neq x_{n+1}$ for all $n \geq 0$.

For $x = x_{2n}$ and $y = x_{2n+1}$ in (2.12), we obtain

$$\begin{aligned}
\psi(d(x_{2n+1}, x_{2n+2})) &= \psi(d(x_{2n+1}, Tx_{2n+1})) \leq \psi(H(Sx_{2n}, Tx_{2n+1})) \\
&\leq \psi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}), \\
&\quad \frac{1}{2}\{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})\}\}) \\
&\quad - \phi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1})\}) \\
&\leq \psi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\
&\quad \frac{1}{2}\{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})\}\}) \\
&\quad - \phi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}) \\
&\leq \psi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\
&\quad \frac{1}{2}\{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})\}\}) \\
&\quad - \phi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}). \tag{2.13}
\end{aligned}$$

If $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$ for some $n \geq 0$, then we have

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq \psi(d(x_{2n+1}, x_{2n+2})) - \phi(d(x_{2n}, x_{2n+2}))$$

which implies $\phi(d(x_{2n}, x_{2n+2})) \leq 0$. Thus we obtain $d(x_{2n}, x_{2n+2}) = 0$, and so $x_{2n} = x_{2n+2}$.

We now show that $x_{2n+1} = x_{2n+2}$.

If $x_{2n+1} \neq x_{2n+2}$, then $0 < d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n+2}, x_{2n}) + d(x_{2n}, x_{2n+1}) = d(x_{2n}, x_{2n+1})$. Thus $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$, which is a contradiction. Thus $x_{2n+1} = x_{2n+2}$. Hence $x_{2n} = x_{2n+1}$, which is a contradiction.

Therefore, $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$ for all $n \geq 0$.

In similar argument, $d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2})$ for all $n \geq 0$.

Thus we have $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \geq 0$, and so $\{d(x_n, x_{n+1})\}$ is a non-decreasing sequence of positive real numbers. Thus there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r.$$

Letting $n \rightarrow \infty$ in (2.13), we obtain $\psi(r) \leq \psi(r) - \phi(r)$. Hence $\phi(r) = 0$, and hence $r = 0$. Thus,

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (2.14)$$

We now show that $\{x_n\}$ is a Cauchy sequence in X .

By (2.14), it sufficient to prove that $\{x_{2n}\}$ is a Cuachy sequence.

Assume that $\{x_{2n}\}$ is not a Cuachy sequence.

Then there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$, there exist $2m(k) > 2n(k) > k$ such that

$$d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon \text{ and } d(x_{2m(k)-1}, x_{2n(k)}) < \epsilon.$$

We have

$$\begin{aligned} \epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \leq d(x_{2m(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2n(k)}) \\ &< d(x_{2m(k)}, x_{2m(k)-1}) + \epsilon. \end{aligned}$$

Letting $k \rightarrow \infty$ in above inequality and using (2.14) we obtain

$$\lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \quad (2.15)$$

Since $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$, we have

$$\lim_{k \rightarrow \infty} d(x_{2m(k)-1}, x_{2n(k)+1}) = \epsilon, \quad (2.16)$$

$$\lim_{k \rightarrow \infty} d(x_{2n(k)}, x_{2m(k)-1}) = \epsilon \quad (2.17)$$

and

$$\lim_{k \rightarrow \infty} d(x_{2n(k)+1}, x_{2m(k)}) = \epsilon. \quad (2.18)$$

And we have

$$\begin{aligned} &d(x_{2n(k)+1}, x_{2m(k)-1}) \\ &\leq d(x_{2n(k)+1}, Tx_{2m(k)-1}) + d(Tx_{2m(k)-1}, x_{2m(k)-1}) \\ &\leq d(x_{2n(k)+1}, Tx_{2m(k)-1}) + d(x_{2m(k)}, x_{2m(k)-1}) \\ &\leq d(x_{2n(k)+1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)-1}). \end{aligned}$$

Taking limit as $k \rightarrow \infty$ in above inequalities and using (2.14), (2.15) and (2.18), we obtain

$$\lim_{n \rightarrow \infty} d(x_{2n(k)+1}, Tx_{2m(k)-1}) = \epsilon. \quad (2.19)$$

For $x = x_{2n(k)}$ and $y = x_{2m(k)-1}$ in (2.12), we have

$$\begin{aligned}
& \psi(d(x_{2n(k)+1}, Tx_{2m(k)-1})) \leq \psi(H(Sx_{2n(k)}, Tx_{2m(k)-1})) \\
& \leq \psi(\max\{d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, Sx_{2n(k)}), d(x_{2m(k)-1}, Tx_{2m(k)-1}), \\
& \quad \frac{1}{2}\{d(x_{2n(k)}, Tx_{2m(k)-1}) + d(x_{2m(k)-1}, Sx_{2n(k)})\}\}) \\
& - \phi(\max\{d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, Sx_{2n(k)}), d(x_{2m(k)-1}, Tx_{2m(k)-1})\}) \\
& \leq \psi(\max\{d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2n(k)+1}), d(x_{2m(k)-1}, x_{2m(k)}), \\
& \quad \frac{1}{2}\{d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)-1}, x_{2n(k)+1})\}\}) \\
& - \phi(\max\{d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2n(k)+1}), d(x_{2m(k)-1}, x_{2m(k)})\})
\end{aligned}$$

Letting $n \rightarrow \infty$ in above inequality and using (2.14), (2.15), (2.16), (2.17) and (2.19), we have $\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon)$, which implies $\phi(\epsilon) = 0$. Thus we have $\epsilon = 0$, which is a contradiction.

Hence $\{x_{2n}\}$ is a Cuachy sequence. By (2.14), $\{x_n\}$ is a Cuachy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$.

For $x = x_{2n}$ and $y = z$ in (2.12), we obtain

$$\begin{aligned}
& \psi(d(x_{2n+1}, Tz)) \leq \psi(H(Sx_{2n}, Tz)) \\
& \leq \psi(\max\{d(x_{2n}, z), d(x_{2n}, Sx_{2n}), d(z, Tz), \\
& \quad \frac{1}{2}\{d(x_{2n}, Tz) + d(z, Sx_{2n})\}\}) \\
& - \phi(\max\{d(x_{2n}, z), d(x_{2n}, Sx_{2n}), d(z, Tz)\}) \\
& \leq \psi(\max\{d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, Tz), \\
& \quad \frac{1}{2}\{d(x_{2n}, Tz) + d(z, x_{2n+1})\}\}) \\
& - \phi(\max\{d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, Tz)\}).
\end{aligned}$$

Letting $n \rightarrow \infty$ in above inequality, we obtain $\psi(d(z, Tz)) \leq \psi(d(z, Tz)) - \phi(d(z, Tz))$, which implies $\phi(d(z, Tz)) = 0$. Hence $d(z, Tz) = 0$, and hence $z \in Tz$. Therefore, z is a common fixed point of S and T . \square

REMARK 2.9. The above result is an extension of theorem 3.2 in [8] to the case set-valued mapping.

From Theorem 2.9 we have the following corollaries.

COROLLARY 2.10. *Let (X, d) be a complete metric space. Suppose that set-valued mappings $S, T : X \rightarrow K(X)$ satisfy*

$$\begin{aligned} \psi(H(Sx, Ty)) &\leq \psi(\max\{d(x, y), \frac{1}{2}\{d(x, Sx) + d(y, Ty)\}, \\ &\frac{1}{2}\{d(x, Ty) + d(y, Sx)\}\}) - \phi(\max\{d(x, y), d(x, Sx), d(y, Ty)\}), \end{aligned}$$

for all $x, y \in X$, where ψ is an altering distance function.

Then S and T have a fixed point in X . Moreover, any fixed point of S is a fixed point of T and conversely.

In Theorem 2.8, if $S = T$, then we have the following corollary.

COROLLARY 2.11. *Let (X, d) be a complete metric space. Suppose that a set-valued mapping $T : X \rightarrow K(X)$ satisfies*

$$\begin{aligned} \psi(H(Tx, Ty)) &\leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), \\ &\frac{1}{2}\{d(x, Ty) + d(y, Tx)\}\}) - \phi(\max\{d(x, y), d(x, Tx), d(y, Ty)\}), \end{aligned}$$

for all $x, y \in X$, where ψ is an altering distance function.

Then T has a fixed point in X .

COROLLARY 2.12. *Let (X, d) be a complete metric space. Suppose that a set-valued mapping $T : X \rightarrow K(X)$ satisfies*

$$\begin{aligned} \psi(H(Tx, Ty)) &\leq \psi(\max\{d(x, y), \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \\ &\frac{1}{2}\{d(x, Ty) + d(y, Tx)\}\}) - \phi(\max\{d(x, y), d(x, Tx), d(y, Ty)\}), \end{aligned}$$

for all $x, y \in X$, where ψ is an altering distance function.

Then T has a fixed point in X .

In Theorem 2.8, if S and T are single valued mappings, then we have the following corollary.

COROLLARY 2.13. [8] *Let (X, d) be a complete metric space. Suppose that mappings $f, g : X \rightarrow X$ satisfy*

$$\begin{aligned} \psi(d(fx, gy)) &\leq \psi(\max\{d(x, y), d(x, fx), d(y, gy), \\ &\frac{1}{2}\{d(x, gy) + d(y, fx)\}\}) - \phi(\max\{d(x, y), d(x, fx), d(y, gy)\}), \quad (2.20) \end{aligned}$$

for all $x, y \in X$, where ψ is an altering distance function.

Then f and g have a unique fixed point in X . Moreover, any fixed point of f is a fixed point of g and conversely.

Proof. By Theorem 2.8, f and g have a common fixed point in X , and any fixed point of f is a fixed point of g and conversely. It follows from (2.20) that the common fixed point of f and g is unique. \square

COROLLARY 2.14. *Let (X, d) be a complete metric space. Suppose that mappings $f, g : X \rightarrow X$ satisfy*

$$\begin{aligned} \psi(d(fx, gy)) &\leq \psi(\max\{d(x, y), \frac{1}{2}\{d(x, fx) + d(y, gy)\}, \\ &\quad \frac{1}{2}\{d(x, gy) + d(y, fx)\}\}) - \phi(\max\{d(x, y), d(x, fx), d(y, gy)\}), \end{aligned}$$

for all $x, y \in X$, where ψ is an altering distance function.

Then f and g have a unique fixed point in X . Moreover, any fixed point of f is a fixed point of g and conversely.

In Corollary 2.14, if g is identical with f , then we have the following corollary.

COROLLARY 2.15. *Let (X, d) be a complete metric space. Suppose that a mapping $f : X \rightarrow X$ satisfies*

$$\begin{aligned} \psi(d(fx, fy)) &\leq \psi(\max\{d(x, y), d(x, fx), d(y, fy), \\ &\quad \frac{1}{2}\{d(x, fy) + d(y, fx)\}\}) - \phi(\max\{d(x, y), d(x, fx), d(y, fy)\}), \end{aligned} \quad (2.21)$$

for all $x, y \in X$, where ψ is an altering distance function.

Then f has a unique fixed point in X .

COROLLARY 2.16. *Let (X, d) be a complete metric space. Suppose that a mapping $f : X \rightarrow X$ satisfies*

$$\begin{aligned} \psi(d(fx, fy)) &\leq \psi(\max\{d(x, y), \frac{1}{2}\{d(x, fx) + d(y, fy)\}, \\ &\quad \frac{1}{2}\{d(x, fy) + d(y, fx)\}\}) - \phi(\max\{d(x, y), d(x, fx), d(y, fy)\}), \end{aligned} \quad (2.22)$$

for all $x, y \in X$, where ψ is an altering distance function.

Then f has a unique fixed point in X .

3. Periodic point theorems

Let X be non-empty set, and let $f : X \rightarrow X$ and let $F(f)$ denote the set of all fixed point of f . We say that f has property P [16] if $F(f) = F(f^n)$ for each $n \in \mathbb{N}$.

THEOREM 3.1. *Let (X, d) be a metric space. Suppose that a mapping $f : X \rightarrow X$ satisfies*

$$\psi(d(fx, f^2x)) < \psi(d(x, fx))$$

for all $x \in X$ with $x \neq fx$, where ψ is an altering distance function.

Then f has property P .

Proof. Suppose that f has no property P . Then there exist $n \in \mathbb{N}$ and $z \in X$ such that $z = f^n z$ and $z \neq fz$. Then we have

$$\begin{aligned} \psi(d(z, fz)) &= \psi(d(f^n z, f^{n+1} z)) \\ &< \psi(d(f^{n-1} z, f^n z)) < \cdots < \psi(d(fz, f^2 z)) \\ &< \psi(d(z, fz)), \end{aligned}$$

which is a contradiction. Thus f has property P . \square

The following proposition is theorem 4.1 in [8]. Here, we give another proof of theorem 4.1 in [8].

PROPOSITION 3.2. *Let (X, d) be a metric space. If a mapping $f : X \rightarrow X$ satisfies (2.9), then f has property P .*

Proof. Let $z \in X$ be such that $z \neq fz$. Then from (2.9) we obtain

$$\begin{aligned} \psi(d(fz, f^2 z)) &\leq \psi(\max\{d(z, fz), d(z, fz), d(fz, f^2 z)\}, \\ &\quad \frac{1}{2}\{d(z, f^2 z) + d(fz, fz)\}) - \phi(\max\{d(z, fz), d(fz, f^2 z)\}). \end{aligned} \quad (3.1)$$

Then $\max\{d(z, fz), d(fz, f^2 z)\} > 0$. In fact, if $\max\{d(z, fz), d(fz, f^2 z)\} = d(z, fz)$ then $\max\{d(z, fz), d(fz, f^2 z)\} = d(z, fz) > 0$ because $z \neq fz$.

Suppose that $\max\{d(z, fz), d(fz, f^2 z)\} = d(fz, f^2 z)$.

If $d(fz, f^2 z) = 0$, then $d(z, fz) = 0$ or $z = fz$, which is a contradiction. Hence $d(fz, f^2 z) > 0$, and hence $\max\{d(z, fz), d(fz, f^2 z)\} > 0$.

Thus, $\phi(\max\{d(z, fz), d(fz, f^2 z)\}) > 0$. Hence from (3.1) we obtain

$$\begin{aligned} &\psi(d(fz, f^2 z)) \\ &< \psi(\max\{d(z, fz), d(z, fz), d(fz, f^2 z)\}, \frac{1}{2}\{d(z, f^2 z) + d(fz, fz)\}) \\ &\leq \psi(\max\{d(z, fz), d(fz, f^2 z)\}, \frac{1}{2}\{d(z, fz) + d(fz, f^2 z)\}) \\ &= \psi(\max\{d(z, fz), d(fz, f^2 z)\}) \\ &= \psi(d(z, fz)). \end{aligned}$$

Thus from Theorem 3.1 f has property P . \square

COROLLARY 3.3. *Let (X, d) be a metric space. If a mapping $f : X \rightarrow X$ satisfies condition (2.10) [or (2.11)], then f has property P .*

THEOREM 3.4. *Let (X, d) be a metric space. If a mapping $f : X \rightarrow X$ satisfies (2.21), then f has property P .*

Proof. Let $z \in X$ be such that $z \neq fz$. Then from (2.21) we obtain

$$\begin{aligned}
& \psi(d(fz, f^2z)) \\
& \leq \psi(\max\{d(z, fz), d(z, fz), d(fz, f^2z), \\
& \quad \frac{1}{2}\{d(z, f^2z) + d(fz, fz)\}\}) \\
& \quad - \phi(\max\{d(z, fz), d(z, fz), d(fz, f^2z)\}) \\
& \leq \psi(\max\{d(z, fz), d(fz, f^2z), \frac{1}{2}\{d(z, fz) + d(fz, f^2z)\}\}) \\
& \quad - \phi(\max\{d(z, fz), d(fz, f^2z)\}) \\
& \leq \psi(\max\{d(z, fz), d(fz, f^2z)\}) - \phi(\max\{d(z, fz), d(fz, f^2z)\}). \quad (3.2)
\end{aligned}$$

As in proof of Proposition 3.2, $\phi(\max\{d(z, fz), d(fz, f^2z)\}) > 0$. Thus from (3.2) we have

$$\begin{aligned}
& \psi(d(fz, f^2z)) < \psi(\max\{d(z, fz), d(fz, f^2z)\}) \\
& = \psi(d(z, fz)).
\end{aligned}$$

Thus from Theorem 3.1 f has property P . \square

COROLLARY 3.5. *Let (X, d) be a metric space. If a mapping $f : X \rightarrow X$ satisfies condition (2.22), then f has property P .*

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