

# GENERALIZED $(\theta, \phi)$ -DERIVATIONS ON POISSON BANACH ALGEBRAS AND JORDAN BANACH ALGEBRAS

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ABSTRACT. In [1], the concept of generalized  $(\theta, \phi)$ -derivations on rings was introduced. In this paper, we introduce the concept of generalized  $(\theta, \phi)$ -derivations on Poisson Banach algebras and of generalized  $(\theta, \phi)$ -derivations on Jordan Banach algebras, and prove the Cauchy–Rassias stability of generalized  $(\theta, \phi)$ -derivations on Poisson Banach algebras and of generalized  $(\theta, \phi)$ -derivations on Jordan Banach algebras.

## 1. Introduction

Let  $X$  and  $Y$  be Banach spaces with norms  $\|\cdot\|$  and  $\|\cdot\|$ , respectively. Consider  $f : X \rightarrow Y$  to be a mapping such that  $f(tx)$  is continuous in  $t \in \mathbb{R}$  for each fixed  $x \in X$ . Th.M. Rassias [10] introduced the following inequality, that we call *Cauchy–Rassias inequality*: Assume that there exist constants  $\epsilon \geq 0$  and  $p \in [0, 1)$  such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all  $x, y \in X$ . Th.M. Rassias [10] showed that there exists a unique  $\mathbb{R}$ -linear mapping  $T : X \rightarrow Y$  such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p$$

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for all  $x \in X$ . Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [2] generalized the Rassias' result in the following form: Let  $G$  be an abelian group and  $X$  a Banach space. Denote by  $\varphi : G \times G \rightarrow [0, \infty)$  a function such that

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x, 2^k y) < \infty$$

for all  $x, y \in G$ . Suppose that  $f : G \rightarrow X$  is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all  $x, y \in G$ . Then there exists a unique additive mapping  $T : G \rightarrow X$  such that

$$\|f(x) - T(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)$$

for all  $x \in G$ .

Jun and Lee [5] proved the following: Denote by  $\varphi : X \setminus \{0\} \times X \setminus \{0\} \rightarrow [0, \infty)$  a function such that

$$\tilde{\varphi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty$$

for all  $x, y \in X \setminus \{0\}$ . Suppose that  $f : X \rightarrow Y$  is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all  $x, y \in X \setminus \{0\}$ . Then there exists a unique additive mapping  $T : X \rightarrow Y$  such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{3} (\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x))$$

for all  $x \in X \setminus \{0\}$ . The stability problem of functional equations has been investigated in several papers (see [8]–[13]).

Throughout this paper, we denote by  $R$  the set of real numbers or complex numbers. Let  $\theta, \phi$  be endomorphisms of  $R$ . An additive mapping  $D : R \rightarrow R$  is called a  $(\theta, \phi)$ -derivation on  $R$  if  $D(xy) = D(x)\theta(y) + \phi(x)D(y)$  holds for all  $x, y \in R$ . An additive mapping  $U : R \rightarrow R$  is called a *generalized*  $(\theta, \phi)$ -derivation on  $R$  if there exists a  $(\theta, \phi)$ -derivation  $D : R \rightarrow R$  such that  $U(xy) = U(x)\theta(y) + \phi(x)D(y)$  holds for all  $x, y \in R$  (see [1], [4]).

In this paper, we are going to introduce the concept of generalized  $(\theta, \phi)$ -derivations on Poisson Banach algebras and of generalized  $(\theta, \phi)$ -derivations on Jordan Banach algebras. We prove the Cauchy–Rassias stability of generalized  $(\theta, \phi)$ -derivations on Poisson Banach algebras and of generalized  $(\theta, \phi)$ -derivations on Jordan Banach algebras.

## 2. GENERALIZED $(\theta, \phi)$ -DERIVATIONS ON POISSON BANACH ALGEBRAS

A *Poisson Banach algebra*  $B$  is a Banach algebra with a  $R$ -bilinear map  $\{\cdot, \cdot\} : B \times B \rightarrow B$ , called a *Poisson bracket*, such that  $(B, \{\cdot, \cdot\})$  is a Lie algebra and

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

for all  $a, b, c \in B$ . Poisson algebras have played an important role in many mathematical areas and have been studied to find symplectic leaves of the corresponding Poisson varieties. It is also important to find or construct a Poisson bracket in the theory of Poisson algebra (see [3], [6], [7]).

Throughout this section, let  $B$  be a Poisson Banach algebra over  $R$  with norm  $\|\cdot\|$ .

DEFINITION 2.1. Let  $\theta, \phi : B \rightarrow B$  be additive mappings. An additive mapping  $D : B \rightarrow B$  is called a  $(\theta, \phi)$ -derivation on  $B$  if  $D(\{x, y\}) = \{D(x), \theta(y)\} + \{\phi(x), D(y)\}$  holds for all  $x, y \in B$ .

An additive mapping  $U : B \rightarrow B$  is called a *generalized*  $(\theta, \phi)$ -derivation on  $B$  if there exists a  $(\theta, \phi)$ -derivation  $D : B \rightarrow B$  such that  $U(\{x, y\}) = \{U(x), \theta(y)\} + \{\phi(x), D(y)\}$  holds for all  $x, y \in B$ .

THEOREM 2.1. Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exists a function  $\varphi : B \times B \rightarrow [0, \infty)$  such that

$$(2.1) \quad \tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty,$$

$$(2.2) \quad \|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y),$$

$$(2.3) \quad \|g(x + y) - g(x) - g(y)\| \leq \varphi(x, y),$$

$$(2.4) \quad \|h(x + y) - h(x) - h(y)\| \leq \varphi(x, y),$$

$$(2.5) \quad \|u(x + y) - u(x) - u(y)\| \leq \varphi(x, y),$$

$$(2.6) \quad \|f(\{x, y\}) - \{f(x), g(y)\} - \{h(x), f(y)\}\| \leq \varphi(x, y),$$

$$(2.7) \quad \|u(\{x, y\}) - \{u(x), g(y)\} - \{h(x), f(y)\}\| \leq \varphi(x, y)$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$(2.8) \quad \|f(x) - D(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x),$$

$$(2.9) \quad \|g(x) - \theta(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x),$$

$$(2.10) \quad \|h(x) - \phi(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x),$$

$$(2.11) \quad \|u(x) - U(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)$$

for all  $x \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* By the Găvruta's theorem [2], it follows from (2.1)–(2.5) that there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  satisfying (2.8)–(2.11). The additive mappings  $D, \theta, \phi, U : B \rightarrow B$  are given by

$$(2.12) \quad D(x) = \lim_{l \rightarrow \infty} \frac{1}{2^l} f(2^l x),$$

$$(2.13) \quad \theta(x) = \lim_{l \rightarrow \infty} \frac{1}{2^l} g(2^l x),$$

$$(2.14) \quad \phi(x) = \lim_{l \rightarrow \infty} \frac{1}{2^l} h(2^l x),$$

$$(2.15) \quad U(x) = \lim_{l \rightarrow \infty} \frac{1}{2^l} u(2^l x),$$

for all  $x \in B$ .

It follows from (2.6) that

$$\begin{aligned} \frac{1}{2^{2l}} \|f(\{2^l x, 2^l y\}) - \{f(2^l x), g(2^l y)\} - \{h(2^l x), f(2^l y)\}\| \\ \leq \frac{1}{2^{2l}} \varphi(2^l x, 2^l y) \\ \leq \frac{1}{2^l} \varphi(2^l x, 2^l y), \end{aligned}$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.1). By (2.12)–(2.14),

$$D(\{x, y\}) = \{D(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all  $x, y \in B$ . So the additive mapping  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ .

It follows from (2.7) that

$$\begin{aligned} \frac{1}{2^{2l}} \|u(\{2^l x, 2^l y\}) - \{u(2^l x), g(2^l y)\} - \{h(2^l x), f(2^l y)\}\| \\ \leq \frac{1}{2^{2l}} \varphi(2^l x, 2^l y) \\ \leq \frac{1}{2^l} \varphi(2^l x, 2^l y), \end{aligned}$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.1). Thus

$$U(\{x, y\}) = \{U(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all  $x, y \in B$ . So the additive mapping  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .  $\square$

**COROLLARY 2.2.** *Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exist constants  $\epsilon \geq 0$  and  $p \in [0, 1)$  such that*

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|g(x+y) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|h(x+y) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(x+y) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|f(\{x, y\}) - \{f(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(\{x, y\}) - \{u(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$\begin{aligned} \|f(x) - D(x)\| &\leq \frac{2\epsilon}{2-2^p} \|x\|^p, \\ \|g(x) - \theta(x)\| &\leq \frac{2\epsilon}{2-2^p} \|x\|^p, \\ \|h(x) - \phi(x)\| &\leq \frac{2\epsilon}{2-2^p} \|x\|^p, \\ \|u(x) - U(x)\| &\leq \frac{2\epsilon}{2-2^p} \|x\|^p \end{aligned}$$

for all  $x \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* Define  $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$  to be Th.M. Rassias upper bound in the Cauchy–Rassias inequality, and apply Theorem 2.1.  $\square$

**THEOREM 2.3.** *Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exists a function  $\varphi : B \times B \rightarrow [0, \infty)$  satisfying (2.6) and (2.7) such that*

$$(2.16) \quad \tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty,$$

$$(2.17) \quad \|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \varphi(x, y),$$

$$(2.18) \quad \|2g(\frac{x+y}{2}) - g(x) - g(y)\| \leq \varphi(x, y),$$

$$(2.19) \quad \|2h(\frac{x+y}{2}) - h(x) - h(y)\| \leq \varphi(x, y),$$

$$(2.20) \quad \|2u(\frac{x+y}{2}) - u(x) - u(y)\| \leq \varphi(x, y)$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$(2.21) \quad \|f(x) - D(x)\| \leq \frac{1}{3} (\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)),$$

$$(2.22) \quad \|g(x) - \theta(x)\| \leq \frac{1}{3} (\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)),$$

$$(2.23) \quad \|h(x) - \phi(x)\| \leq \frac{1}{3} (\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)),$$

$$(2.24) \quad \|u(x) - U(x)\| \leq \frac{1}{3} (\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x))$$

for all  $x \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* By the Jun and Lee's theorem [5, Theorem 1], it follows from (2.16)–(2.20) that there exist unique additive mappings  $D, \theta, \phi, U :$

$B \rightarrow B$  satisfying (2.21)–(2.24). The additive mappings  $D, \theta, \phi, U : B \rightarrow B$  are given by

$$(2.25) \quad D(x) = \lim_{l \rightarrow \infty} \frac{1}{3^l} f(3^l x),$$

$$(2.26) \quad \theta(x) = \lim_{l \rightarrow \infty} \frac{1}{3^l} g(3^l x),$$

$$(2.27) \quad \phi(x) = \lim_{l \rightarrow \infty} \frac{1}{3^l} h(3^l x),$$

$$(2.28) \quad U(x) = \lim_{l \rightarrow \infty} \frac{1}{3^l} u(3^l x),$$

for all  $x \in B$ .

It follows from (2.6) that

$$\begin{aligned} \frac{1}{3^{2l}} \|f(\{3^l x, 3^l y\}) - \{f(3^l x), g(3^l y)\} - \{h(3^l x), f(3^l y)\}\| \\ \leq \frac{1}{3^{2l}} \varphi(3^l x, 3^l y) \\ \leq \frac{1}{3^l} \varphi(3^l x, 3^l y), \end{aligned}$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.16). By (2.25)–(2.28),

$$D(\{x, y\}) = \{D(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all  $x, y \in B$ . So the additive mapping  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ .

It follows from (2.7) that

$$\begin{aligned} \frac{1}{3^{2l}} \|u(\{3^l x, 3^l y\}) - \{u(3^l x), g(3^l y)\} - \{h(3^l x), f(3^l y)\}\| \\ \leq \frac{1}{3^{2l}} \varphi(3^l x, 3^l y) \\ \leq \frac{1}{3^l} \varphi(3^l x, 3^l y), \end{aligned}$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.16). Thus

$$U(\{x, y\}) = \{U(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all  $x, y \in B$ . So the additive mapping  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .  $\square$

**COROLLARY 2.4.** *Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exist constants  $\epsilon \geq 0$  and  $p \in [0, 1)$  such that*

$$\begin{aligned} \|2f(\frac{x+y}{2}) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2g(\frac{x+y}{2}) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2h(\frac{x+y}{2}) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2u(\frac{x+y}{2}) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|f(\{x, y\}) - \{f(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(\{x, y\}) - \{u(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$\begin{aligned} \|f(x) - D(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\ \|g(x) - \theta(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\ \|h(x) - \phi(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\ \|u(x) - U(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p \end{aligned}$$

for all  $x \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* Define  $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ , and apply Theorem 2.3.  $\square$

THEOREM 2.5. Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exists a function  $\varphi : B \times B \rightarrow [0, \infty)$  satisfying (2.17)–(2.20), (2.6) and (2.7) such that

$$(2.29) \quad \sum_{j=0}^{\infty} 3^{2j} \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right) < \infty$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$(2.30) \quad \|f(x) - D(x)\| \leq \tilde{\varphi}\left(\frac{x}{3}, -\frac{x}{3}\right) + \tilde{\varphi}\left(-\frac{x}{3}, x\right),$$

$$(2.31) \quad \|g(x) - \theta(x)\| \leq \tilde{\varphi}\left(\frac{x}{3}, -\frac{x}{3}\right) + \tilde{\varphi}\left(-\frac{x}{3}, x\right),$$

$$(2.32) \quad \|h(x) - \phi(x)\| \leq \tilde{\varphi}\left(\frac{x}{3}, -\frac{x}{3}\right) + \tilde{\varphi}\left(-\frac{x}{3}, x\right),$$

$$(2.33) \quad \|u(x) - U(x)\| \leq \tilde{\varphi}\left(\frac{x}{3}, -\frac{x}{3}\right) + \tilde{\varphi}\left(-\frac{x}{3}, x\right)$$

for all  $x \in B$ , where

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 3^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right)$$

for all  $x, y \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* Note that  $\sum_{j=0}^{\infty} 3^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right) \leq \sum_{j=0}^{\infty} 3^{2j} \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right)$  for all  $x, y \in B$ . By the Jun and Lee's theorem [5, Theorem 6], it follows from (2.29) and (2.17)–(2.20) that there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  satisfying (2.30)–(2.33). The additive mappings  $D, \theta, \phi, U : B \rightarrow B$  are given by

$$(2.34) \quad D(x) = \lim_{l \rightarrow \infty} 3^l f\left(\frac{x}{3^l}\right),$$

$$(2.35) \quad \theta(x) = \lim_{l \rightarrow \infty} 3^l g\left(\frac{x}{3^l}\right),$$

$$(2.36) \quad \phi(x) = \lim_{l \rightarrow \infty} 3^l h\left(\frac{x}{3^l}\right),$$

$$(2.37) \quad U(x) = \lim_{l \rightarrow \infty} 3^l u\left(\frac{x}{3^l}\right),$$

for all  $x \in B$ .

It follows from (2.6) that

$$3^{2l} \|f(\{\frac{x}{3^l}, \frac{y}{3^l}\}) - \{f(\frac{x}{3^l}), g(\frac{y}{3^l})\} - \{h(\frac{x}{3^l}), f(\frac{y}{3^l})\}\| \leq 3^{2l} \varphi(\frac{x}{3^l}, \frac{y}{3^l}),$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.29). By (2.34)–(2.37),

$$D(\{x, y\}) = \{D(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all  $x, y \in B$ . So the additive mapping  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ .

It follows from (2.7) that

$$3^{2l} \|u(\{\frac{x}{3^l}, \frac{y}{3^l}\}) - \{u(\frac{x}{3^l}), g(\frac{y}{3^l})\} - \{h(\frac{x}{3^l}), f(\frac{y}{3^l})\}\| \leq 3^{2l} \varphi(\frac{x}{3^l}, \frac{y}{3^l}),$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.29). Thus

$$U(\{x, y\}) = \{U(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all  $x, y \in B$ . So the additive mapping  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .  $\square$

**COROLLARY 2.6.** *Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exist constants  $\epsilon \geq 0$  and  $p \in (2, \infty)$  such that*

$$\begin{aligned} \|2f(\frac{x+y}{2}) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2g(\frac{x+y}{2}) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2h(\frac{x+y}{2}) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2u(\frac{x+y}{2}) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|f(\{x, y\}) - \{f(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(\{x, y\}) - \{u(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$\begin{aligned}\|f(x) - D(x)\| &\leq \frac{3^p + 3}{3^p - 3} \epsilon \|x\|^p, \\ \|g(x) - \theta(x)\| &\leq \frac{3^p + 3}{3^p - 3} \epsilon \|x\|^p, \\ \|h(x) - \phi(x)\| &\leq \frac{3^p + 3}{3^p - 3} \epsilon \|x\|^p, \\ \|u(x) - U(x)\| &\leq \frac{3^p + 3}{3^p - 3} \epsilon \|x\|^p\end{aligned}$$

for all  $x \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* Define  $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ , and apply Theorem 2.5.  $\square$

*Remark 2.1.* A Banach algebra  $B$ , endowed with the Lie product  $[x, y] = \frac{xy - yx}{2}$  on  $B$ , is called a *Lie Banach algebra*. Let  $\theta, \phi : B \rightarrow B$  be additive mappings. An additive mapping  $D : B \rightarrow B$  is called a  $(\theta, \phi)$ -derivation on a Lie Banach algebra  $B$  if  $D([x, y]) = [D(x), \theta(y)] + [\phi(x), D(y)]$  holds for all  $x, y \in B$ . An additive mapping  $U : B \rightarrow B$  is called a *generalized  $(\theta, \phi)$ -derivation* on a Lie Banach algebra  $B$  if there exists a  $(\theta, \phi)$ -derivation  $D : B \rightarrow B$  such that  $U([x, y]) = [U(x), \theta(y)] + [\phi(x), D(y)]$  holds for all  $x, y \in B$ .

When each Poisson bracket  $\{\cdot, \cdot\}$  in this section is replaced by the Lie product  $[\cdot, \cdot]$ , we can show that there exists a unique  $(\theta, \phi)$ -derivation  $D : B \rightarrow B$  on a Lie Banach algebra, and there exists a unique generalized  $(\theta, \phi)$ -derivation  $U : B \rightarrow B$  on a Lie Banach algebra.

### 3. GENERALIZED $(\theta, \phi)$ -DERIVATIONS ON JORDAN BANACH ALGEBRAS

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see

[14]). Let  $\mathcal{L}(\mathcal{H})$  be the real vector space of all bounded self-adjoint linear operators on  $\mathcal{H}$ , interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that  $\mathcal{L}(\mathcal{H})$  is a (nonassociative) algebra via the *anticommutator product*  $x \circ y := \frac{xy+yx}{2}$ . A commutative Banach algebra  $X$  with product  $x \circ y$  is called a *Jordan Banach algebra*.

Throughout this section, let  $B$  be a Jordan Banach algebra over  $R$  with norm  $\| \cdot \|$ .

DEFINITION 3.1. Let  $\theta, \phi : B \rightarrow B$  be additive mappings. An additive mapping  $D : B \rightarrow B$  is called a  $(\theta, \phi)$ -*derivation* on  $B$  if  $D(x \circ y) = D(x) \circ \theta(y) + \phi(x) \circ D(y)$  holds for all  $x, y \in B$ .

An additive mapping  $U : B \rightarrow B$  is called a *generalized  $(\theta, \phi)$ -derivation* on  $B$  if there exists a  $(\theta, \phi)$ -derivation  $D : B \rightarrow B$  such that  $U(x \circ y) = U(x) \circ \theta(y) + \phi(x) \circ D(y)$  holds for all  $x, y \in B$ .

THEOREM 3.1. Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exists a function  $\varphi : B \times B \rightarrow [0, \infty)$  satisfying (2.1)–(2.5) such that

$$(3.1) \quad \|f(x \circ y) - f(x) \circ g(y) - h(x) \circ f(y)\| \leq \varphi(x, y),$$

$$(3.2) \quad \|u(x \circ y) - u(x) \circ g(y) - h(x) \circ f(y)\| \leq \varphi(x, y)$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  satisfying (2.8)–(2.11). Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* By the Găvruta's theorem [3], it follows from (2.1)–(2.5) that there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  satisfying (2.8)–(2.11).

It follows from (3.1) that

$$\begin{aligned} \frac{1}{2^{2l}} \|f((2^l x) \circ (2^l y)) - f(2^l x) \circ g(2^l y) - h(2^l x) \circ f(2^l y)\| \\ \leq \frac{1}{2^{2l}} \varphi(2^l x, 2^l y) \\ \leq \frac{1}{2^l} \varphi(2^l x, 2^l y), \end{aligned}$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.1). By (2.12)–(2.15),

$$D(x \circ y) = D(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all  $x, y \in B$ . So the additive mapping  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ .

It follows from (3.2) that

$$\begin{aligned} \frac{1}{2^{2l}} \|u((2^l x) \circ (2^l y)) - u(2^l x) \circ g(2^l y) - h(2^l x) \circ f(2^l y)\| \\ \leq \frac{1}{2^{2l}} \varphi(2^l x, 2^l y) \\ \leq \frac{1}{2^l} \varphi(2^l x, 2^l y), \end{aligned}$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.1). Thus

$$U(x \circ y) = U(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all  $x, y \in B$ . So the additive mapping  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .  $\square$

**COROLLARY 3.2.** *Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exist constants  $\epsilon \geq 0$  and  $p \in [0, 1)$  such that*

$$\begin{aligned} \|f(x + y) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|g(x + y) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|h(x + y) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(x + y) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \end{aligned}$$

$$\begin{aligned}\|f(x \circ y) - f(x) \circ g(y) - h(x) \circ f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(x \circ y) - u(x) \circ g(y) - h(x) \circ f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p)\end{aligned}$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$\begin{aligned}\|f(x) - D(x)\| &\leq \frac{2\epsilon}{2-2^p}\|x\|^p, \\ \|g(x) - \theta(x)\| &\leq \frac{2\epsilon}{2-2^p}\|x\|^p, \\ \|h(x) - \phi(x)\| &\leq \frac{2\epsilon}{2-2^p}\|x\|^p, \\ \|u(x) - U(x)\| &\leq \frac{2\epsilon}{2-2^p}\|x\|^p\end{aligned}$$

for all  $x \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* Define  $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$  to be Th.M. Rassias upper bound in the Cauchy–Rassias inequality, and apply Theorem 3.1.  $\square$

**THEOREM 3.3.** *Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exists a function  $\varphi : B \times B \rightarrow [0, \infty)$  satisfying (3.1), (3.2) and (2.16)–(2.20). Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  satisfying (2.21)–(2.24). Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .*

*Proof.* By the Jun and Lee’s theorem [5, Theorem 1], it follows from (2.16)–(2.20) that there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  satisfying (2.21)–(2.24).

It follows from (3.1) that

$$\begin{aligned}\frac{1}{3^{2l}}\|f((3^l x) \circ (3^l y)) - f(3^l x) \circ g(3^l y) - h(3^l x) \circ f(3^l y)\| \\ \leq \frac{1}{3^{2l}}\varphi(3^l x, 3^l y) \leq \frac{1}{3^l}\varphi(3^l x, 3^l y),\end{aligned}$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.16). By (2.25)–(2.28),

$$D(x \circ y) = D(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all  $x, y \in B$ . So the additive mapping  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ .

It follows from (3.2) that

$$\begin{aligned} \frac{1}{3^{2l}} \|u((3^l x) \circ (3^l y)) - u(3^l x) \circ g(3^l y) - h(3^l x) \circ f(3^l y)\| \\ \leq \frac{1}{3^{2l}} \varphi(3^l x, 3^l y) \\ \leq \frac{1}{3^l} \varphi(3^l x, 3^l y), \end{aligned}$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.16). Thus

$$U(x \circ y) = U(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all  $x, y \in B$ . So the additive mapping  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .  $\square$

**COROLLARY 3.4.** *Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exist constants  $\epsilon \geq 0$  and  $p \in [0, 1)$  such that*

$$\begin{aligned} \|2f(\frac{x+y}{2}) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2g(\frac{x+y}{2}) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2h(\frac{x+y}{2}) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2u(\frac{x+y}{2}) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|f(x \circ y) - f(x) \circ g(y) - h(x) \circ f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(x \circ y) - u(x) \circ g(y) - h(x) \circ f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$\begin{aligned}\|f(x) - D(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\ \|g(x) - \theta(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\ \|h(x) - \phi(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\ \|u(x) - U(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p\end{aligned}$$

for all  $x \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* Define  $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ , and apply Theorem 3.3.  $\square$

**THEOREM 3.5.** Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exists a function  $\varphi : B \times B \rightarrow [0, \infty)$  satisfying (2.17)–(2.20), (3.1), (3.2) and (2.29). Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  satisfying (2.30)–(2.33). Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* By the Jun and Lee's theorem [5, Theorem 6], it follows from (2.29) and (2.17)–(2.20) that there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  satisfying (2.30)–(2.33).

It follows from (3.1) that

$$3^{2l} \|f(\frac{x}{3^l} \circ \frac{y}{3^l}) - f(\frac{x}{3^l}) \circ g(\frac{y}{3^l}) - h(\frac{x}{3^l}) \circ f(\frac{y}{3^l})\| \leq 3^{2l} \varphi(\frac{x}{3^l}, \frac{y}{3^l}),$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.29). By (2.34)–(2.37),

$$D(x \circ y) = D(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all  $x, y \in B$ . So the additive mapping  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ .

It follows from (3.2) that

$$3^{2l} \|u(\frac{x}{3^l} \circ \frac{y}{3^l}) - u(\frac{x}{3^l}) \circ g(\frac{y}{3^l}) - h(\frac{x}{3^l}) \circ f(\frac{y}{3^l})\| \leq 3^{2l} \varphi(\frac{x}{3^l}, \frac{y}{3^l}),$$

which tends to zero as  $l \rightarrow \infty$  for all  $x, y \in B$  by (2.29). Thus

$$U(x \circ y) = U(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all  $x, y \in B$ . So the additive mapping  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .  $\square$

**COROLLARY 3.6.** *Let  $f, g, h, u : B \rightarrow B$  be mappings with  $f(0) = g(0) = h(0) = u(0) = 0$  for which there exist constants  $\epsilon \geq 0$  and  $p \in (2, \infty)$  such that*

$$\begin{aligned} \|2f(\frac{x+y}{2}) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2g(\frac{x+y}{2}) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2h(\frac{x+y}{2}) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2u(\frac{x+y}{2}) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|f(x \circ y) - f(x) \circ g(y) - h(x) \circ f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(x \circ y) - u(x) \circ g(y) - h(x) \circ f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all  $x, y \in B$ . Then there exist unique additive mappings  $D, \theta, \phi, U : B \rightarrow B$  such that

$$\begin{aligned} \|f(x) - D(x)\| &\leq \frac{3^p + 3}{3^p - 3} \epsilon \|x\|^p, \\ \|g(x) - \theta(x)\| &\leq \frac{3^p + 3}{3^p - 3} \epsilon \|x\|^p, \\ \|h(x) - \phi(x)\| &\leq \frac{3^p + 3}{3^p - 3} \epsilon \|x\|^p, \\ \|u(x) - U(x)\| &\leq \frac{3^p + 3}{3^p - 3} \epsilon \|x\|^p \end{aligned}$$

for all  $x \in B$ . Moreover,  $D : B \rightarrow B$  is a  $(\theta, \phi)$ -derivation on  $B$ , and  $U : B \rightarrow B$  is a generalized  $(\theta, \phi)$ -derivation on  $B$ .

*Proof.* Define  $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ , and apply Theorem 3.5.  $\square$

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