JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **26**, No. 4, November 2013
http://dx.doi.org/10.14403/jcms.2013.26.4.887

STABILITY OF A GENERALIZED POLYNOMIAL FUNCTIONAL EQUATION OF DEGREE 2 IN NON-ARCHIMEDEAN NORMED SPACES

Chang-Ju Lee* and Yang-Hi Lee**

Abstract. In this paper, we investigate the stability for the functional equation

$$f(3x+y)-3f(2x+y)+3f(x+y)-f(y)=0$$

in the sense of M. S. Moslehian and Th. M. Rassias.

1. Introduction

The stability problem of the functional equation was formulated by S. M. Ulam [16] in 1940. D. H. Hyers [4], T. Aoki [1] and Th. M. Rassias [15] made important role to study the stability of the functional equation. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [2],[3], [6]-[13].

By *a non-Archimedean field*, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$, and $|r+s|\le \max\{|r|,|s|\}$ for all $r,s\in\mathbb{K}$. Clearly $|1|=|-1|$ and $|n|\le 1$ for all $n\in\mathbb{N}$. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non- trivial valuation $|\cdot|$. A function $\|\cdot\|: X\to\mathbb{R}$ is *a non-Archimedean norm (valuation)* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;
(ii) $\|rx\|=|r|\|x\|$ $(r\in\mathbb{K}, x\in X)$;

Received September 24, 2013; Accepted November 04, 2013.
2010 Mathematics Subject Classification: Primary 46S40, 39B52.
Key words and phrases: generalized polynomial functional equation of degree 2, non-Archimedean normed space, stability of functional equation.
Correspondence should be addressed to Yang-Hi Lee, lyhmzi@gjue.ac.kr.
This work was supported by Gongju National University of Education Grant.

(iii) the strong triangle inequality, namely,

$$\|x+y\| \le \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$ and $r \in \mathbb{K}$.

Then $(X, \|\cdot\|)$ is called *a non-Archimedean normed space.* Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\| : m \le j \le n-1\} (n > m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By *a complete non-Archimedean space,* we mean one in which every Cauchy sequence is convergent.

Recently M. S. Moslehian and Th. M. Rassias [14] discussed the Hyers-Ulam stability of the Cauchy functional equation $f(x+y) = f(x) + f(y)$ and the quadratic functional equation $f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0$ in non-Archimedean normed spaces.

Now we consider the *generalized polynomial functional equation of degree 2*

$$f(3x+y) - 3f(2x+y) + 3f(x+y) - f(y) = 0$$

whose solution is called *a general quadratic mapping.* In 2009, the second author [9] obtained a stability of the generalized polynomial functional equation of degree 2 by taking and composing an additive mapping A and a quadratic mapping Q to prove the existence of a general quadratic function F which is close to the given function f. In his processing, A is approximate to the odd part $\frac{f(x)-f(-x)}{2}$ of f and Q is close to the even part $\frac{f(x)+f(-x)}{2} - f(0)$ of it, respectively.

In this paper, we get a general stability result of the generalized polynomial functional equation of degree 2 in non-Archimedean normed spaces.

2. Stability of the generalized polynomial functional equation of degree 2

In this section, we prove the generalized Hyers-Ulam stability of the generalized polynomial functional equation of degree 2. Throughout this section, we assume that X is a non-Archimedean normed space and Y is a complete non-Archimedean space.

For a given mapping $f : X \to Y$, we use the abbreviation

$$Df(x,y) := f(3x+y) - 3f(2x+y) + 3f(x+y) - f(y)$$

for all $x, y \in X$.

LEMMA 2.1. (Lemma 3.1 in [5]) *If $f : X \to Y$ is a mapping such that $Df(x, y) = 0$ for all $x, y \in X\backslash\{0\}$, then f is a general quadratic mapping.*

THEOREM 2.2. *Let $\varphi : (X\backslash\{0\})^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|4|^n} = 0 \tag{2.1}$$

for all $x, y \in X\backslash\{0\}$ and let for each $x \in X\backslash\{0\}$ the limit

$$\lim_{n\to\infty} \max_{0\le j<n} \left\{ \frac{\varphi(2^j x, -2^j x)}{|2|\cdot|4|^{j+1}}, \frac{\varphi(-2^j x, 2^j x)}{|2|\cdot|4|^{j+1}} \right\}, \tag{2.2}$$

denoted by $\tilde{\varphi}(x)$, exists. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

$$\|Df(x, y)\| \le \varphi(x, y) \tag{2.3}$$

for all $x, y \in X\backslash\{0\}$. Then there exists a unique general quadratic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \tilde{\varphi}(x) \tag{2.4}$$

for all $x \in X\backslash\{0\}$ with $T(0) = f(0)$. In particular, T is given by

$$\begin{aligned} T(x) &= \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x) - 2f(0)}{2\cdot 4^n} \\ &\quad + \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} + f(0) \end{aligned}$$

for all $x \in X$.

Proof. Let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x) - 2f(0)}{2\cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} + f(0)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and

$$\begin{aligned}\|J_jf(x)-J_{j+1}f(x)\| &= \Big\|-\frac{Df(2^jx,-2^jx)}{2\cdot 4^{j+1}}-\frac{Df(-2^jx,2^jx)}{2\cdot 4^{j+1}}\\ &\quad -\frac{Df(2^jx,-2^jx)}{2^{j+2}}+\frac{Df(-2^jx,2^jx)}{2^{j+2}}\Big\|\\ &\le \max\Big\{\frac{\|Df(2^jx,-2^jx)\|}{|2|\cdot|4|^{j+1}},\frac{\|Df(-2^jx,2^jx)\|}{|2|\cdot|4|^{j+1}},\\ &\qquad \frac{\|Df(2^jx,-2^jx)\|}{|2|^{j+2}},\frac{\|Df(-2^jx,2^jx)\|}{|2|^{j+2}}\Big\}\\ &\le \max\Big\{\frac{\varphi(2^jx,-2^jx)}{|2|\cdot|4|^{j+1}},\frac{\varphi(-2^jx,2^jx)}{|2|\cdot|4|^{j+1}}\Big\} \qquad (2.5)\end{aligned}$$

for all $x \in X\backslash\{0\}$ and all $j \geq 0$. It follows from (2.5) and (2.1) that the sequence $\{J_nf(x)\}$ is Cauchy for all $x \in X\backslash\{0\}$. Since Y is complete and $J_nf(0) = f(0)$ for all $n \in \mathbb{N}$, we conclude that $\{J_nf(x)\}$ is convergent for all $x \in X$. Set

$$T(x) := \lim_{n\to\infty} J_nf(x).$$

One can show that

$$\begin{aligned}\|J_nf(x)-f(x)\| &= \Big\|\sum_{j=0}^{n-1} J_jf(x)-J_{j+1}f(x)\Big\|\\ &\le \max_{0\le j<n}\Big\{\frac{\varphi(2^jx,-2^jx)}{|2|\cdot|4|^{j+1}},\frac{\varphi(-2^jx,2^jx)}{|2|\cdot|4|^{j+1}}\Big\} \qquad (2.6)\end{aligned}$$

for all $n \in N$ and all $x \in X\backslash\{0\}$. By taking n to approach infinity in (2.6) and using (2.2) one obtains (2.4). Replacing x and y by 2^nx and 2^ny, respectively, in (2.3) we get

$$\begin{aligned}\|DJ_nf(x,y)\| &= \Big\|\frac{Df(2^nx,2^ny)-Df(-2^nx,-2^ny)}{2^{n+1}}\\ &\quad +\frac{Df(2^nx,2^ny)+Df(-2^nx,-2^ny)}{2^{2n+1}}\Big\|\\ &\le \max\Big\{\frac{\varphi(2^nx,2^ny)}{|2|^{n+1}},\frac{\varphi(-2^nx,-2^ny)}{|2|^{n+1}},\\ &\qquad \frac{\varphi(2^nx,2^ny)}{|2|\cdot|4|^n},\frac{\varphi(-2^nx,-2^ny)}{|2|\cdot|4|^n}\Big\}\end{aligned}$$

for all $x, y \in X\backslash\{0\}$ and all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and using (2.1) and Lemma 2.1 we get $DT(x,y) = 0$ for all $x, y \neq 0$ and so T is a general quadratic mapping. Now we are going to prove the uniqueness of T. If T' is another general quadratic mapping satisfying (2.4) with $T'(0) = f(0)$, then

$$\begin{aligned} T'(x) &= \sum_{j=0}^{k-1}\Bigg(-\frac{DT'(2^jx,-2^jx)}{2\cdot 4^{j+1}} - \frac{DT'(-2^jx,2^jx)}{2\cdot 4^{j+1}} \\ &\quad -\frac{DT'(2^jx,-2^jx)}{2^{j+2}} + \frac{DT'(-2^jx,2^jx)}{2^{j+2}}\Bigg) + J_kT'(x) \\ &= J_kT'(x) \end{aligned}$$

for any $k \in N$ and so

$$\begin{aligned} &\|T(x) - T'(x)\| \\ &= \lim_{k\to\infty} \|J_kT(x) - J_kT'(x)\| \\ &\le \lim_{k\to\infty} \max\{\|J_kT(x) - J_kf(x)\|, \|J_kf(x) - J_kT'(x)\|\} \\ &\le \lim_{k\to\infty} |2|^{-2k-1}\max\{\|T(2^kx) - f(2^kx)\|, \|T(-2^kx) - f(-2^kx)\|, \\ &\qquad\qquad \|f(2^kx) - T'(2^kx)\|, \|f(-2^kx) - T'(-2^kx)\|\} \\ &\le \lim_{k\to\infty}\lim_{n\to\infty}\max_{k\le j<n+k}\left\{\frac{\varphi(2^jx,-2^jx)}{|4|^{j+2}}, \frac{\varphi(-2^jx,2^jx)}{|4|^{j+2}}\right\} \\ &= 0 \end{aligned}$$

for all $x \in X\backslash\{0\}$. Since $T(0) = f(0) = T'(0)$, we get $T(x) = T'(x)$ for all $x \in X$. This completes the proof of the uniqueness of T. □

Corollary 2.3. *Let $2 < r$ be a real number and $|2| < 1$. If $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \le \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$, then there exists a unique general quadratic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le 2\theta|2|^{-3}\|x\|^r \tag{2.7}$$

for all $x \in X$ with $T(0) = f(0)$.

Proof. Let $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$. Since $|2| < 1$ and $r - 2 > 0$,

$$\lim_{n\to\infty} |4|^{-n}\varphi(2^nx, 2^ny) = \lim_{n\to\infty} |2|^{n(r-2)}\varphi(x,y) = 0$$

for all $x, y \in X$. Therefore the conditions of Theorem 2.2 are fulfilled and it is easy to see that $\tilde{\varphi}(x) = 2\theta|2|^{-3}\|x\|^r$. By Theorem 2.2 there is a unique general quadratic mapping $T : X \to Y$ satisfying (2.7) with $T(0) = f(0)$. $\square$

THEOREM 2.4. *Let $\varphi : (X\backslash\{0\})^2 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \varphi(2^{-n}x, 2^{-n}y) = 0 \tag{2.8}$$

for all $x, y \in X\backslash\{0\}$ and let for each $x \in X\backslash\{0\}$ the limit

$$\lim_{n\to\infty} \max_{0\le j<n} \left\{ |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), |2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right\}, \tag{2.9}$$

denoted by $\tilde{\varphi}(x)$, exists. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.10}$$

for all $x, y \in X\backslash\{0\}$. Then there exists a unique general quadratic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \tilde{\varphi}(x) \tag{2.11}$$

for all $x \in X\backslash\{0\}$ with $T(0) = f(0)$. In particular, T is given by

$$\begin{aligned} T(x) &= \lim_{n\to\infty} \frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right) + f\left(\frac{-x}{2^n}\right) - 2f(0)\right) \\ &\quad + \lim_{n\to\infty} 2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right)\right) + f(0) \end{aligned}$$

for all $x \in X$.

Proof. Let $J_n f : X \to Y$ be a mapping defined by

$$\begin{aligned} J_n f(x) &= \frac{4^n}{2}\left(f(2^{-n}x) + f(-2^{-n}x) - 2f(0)\right) \\ &\quad + 2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right) + f(0) \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and

$$\begin{aligned} &\|J_j f(x) - J_{j+1} f(x)\| \\ &\quad = \left\| (2^{2j-1} + 2^{j-1}) Df\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right) \right. \\ &\qquad \left. + (2^{2j-1} - 2^{j-1}) Df\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right\| \\ &\quad \le \max\left\{ |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), |2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right\} \qquad (2.12) \end{aligned}$$

for all $x \in X\backslash\{0\}$ and all $j \geq 0$. It follows from (2.12) and (2.8) that the sequence $\{J_n f(x)\}$ is Cauchy for all $x \in X\backslash\{0\}$. Since Y is complete and $J_n f(0) = f(0)$, we conclude that $\{J_n f(x)\}$ is convergent for all $x \in X$. Set

$$T(x) := \lim_{n\to\infty} J_n f(x)$$

for all $x \in X$. Using induction one can show that

$$\begin{aligned}
&\|J_n f(x) - f(x)\| \\
&\leq \max_{0\leq j<n} \left\{ |2|^{j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), |2|^{j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right\}
\end{aligned} \tag{2.13}$$

for all $n \in N$ and all $x \in X\backslash\{0\}$. By taking n to approach infinity in (2.13) and using (2.9) one obtains (2.11). Replacing x and y by $2^{-n}x$ and $2^{-n}y$, respectively, in (2.10) we get

$$\begin{aligned}
\|DJ_n f(x,y)\| &= \left\| 2^{n-1}Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - 2^{n-1}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right. \\
&\quad \left. +2^{2n-1}Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 2^{2n-1}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right\| \\
&\leq \max\left\{ |2|^{n-1}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right), |2|^{n-1}\varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right\}
\end{aligned}$$

for all $x, y \in X\backslash\{0\}$. Taking the limit as $n \to \infty$ and using (2.8) and Lemma 2.1 we get $DT(x,y) = 0$ for all $x, y \neq 0$ and so T is a general quadratic mapping. Now we are going to prove the uniqueness of T. If T' is another general quadratic mapping satisfying (2.11) with $T'(0) = f(0)$, then

$$\begin{aligned}
T'(x) - J_k T'(x) &= \sum_{j=0}^{k-1} \left((2^{2j-1} + 2^{j-1})DT'\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right) \right. \\
&\quad \left. +(2^{2j-1} - 2^{j-1})DT'\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right) \\
&= 0
\end{aligned}$$

for any $k \in \mathbb{N}$ and so

$$
\begin{aligned}
&\|T(x) - T'(x)\| \\
&= \lim_{k\to\infty} \|J_kT(x) - J_kT'(x)\| \\
&\leq \lim_{k\to\infty} \max\{\|J_kT(x) - J_kf(x)\|, \|J_kf(x) - J_kT'(x)\|\} \\
&\leq \lim_{k\to\infty} |2|^{k-1} \max\left\{ \left\|T\left(\frac{x}{2^k}\right) - f\left(\frac{x}{2^k}\right)\right\|, \left\|T\left(-\frac{x}{2^k}\right) - f\left(-\frac{x}{2^k}\right)\right\|,\right. \\
&\qquad\qquad \left. \left\|f\left(\frac{x}{2^k}\right) - T'\left(\frac{x}{2^k}\right)\right\|, \left\|f\left(-\frac{x}{2^k}\right) - T'\left(-\frac{x}{2^k}\right)\right\| \right\} \\
&\leq \lim_{k\to\infty} |2|^{k-1}\tilde{\varphi}\left(\frac{x}{2^k}\right) \\
&= 0
\end{aligned}
$$

for all $x \in X\backslash\{0\}$. Since $T(0 = f(0) = T'(0)$, we get $T(x) = T'(x)$ for all $x \in X$. This completes the proof of the uniqueness of T.

□

Corollary 2.5. *Let $r < 1$ be a real number and $|2| < 1$. If $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X\backslash\{0\}$, then there exists a unique general quadratic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq 2\theta|2|^{-1-r}\|x\|^r \tag{2.14}$$

for all $x \in X\backslash\{0\}$ with $T(0) = f(0)$.

Proof. Let $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$. Since $|2| < 1$ and $1 - r > 0$, we get

$$\lim_{n\to\infty} |2|^n\varphi(2^{-n}x, 2^{-n}y) = \lim_{n\to\infty} |2|^{n(1-r)}\varphi(x,y) = 0$$

for all $x, y \in X\backslash\{0\}$. Therefore the conditions of Theorem 2.4 are fulfilled and it is easy to see that $\tilde{\varphi}(x) = 2\theta|2|^{-1-r}\|x\|^r$. By Theorem 2.4 there is a unique general quadratic mapping $T : X \to Y$ satisfying (2.14) with $T(0) = f(0)$. □

Theorem 2.6. *Let $\varphi : (X\backslash\{0\})^2 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} |4|^n\varphi(2^{-n}x, 2^{-n}y) = 0 \tag{2.15}$$

and

$$\lim_{n\to\infty} \frac{\varphi(2^nx, 2^ny)}{|2|^n} = 0 \tag{2.16}$$

for all $x, y \in X\backslash\{0\}$ and let for each $x \in X\backslash\{0\}$ the limit

$$\lim_{n\to\infty} \max_{0\le j<n} \left\{ |2|^{2j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), |2|^{2j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \right.$$

$$\left. \frac{\varphi(2^j x, -2^j x)}{|2|^{j+2}}, \frac{\varphi(-2^j x, 2^j x)}{|2|^{j+2}} \right\}, \tag{2.17}$$

denoted by $\tilde{\varphi}(x)$, exists. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.18}$$

for all $x, y \in X\backslash\{0\}$. Then there exists a unique general quadratic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \tilde{\varphi}(x) \tag{2.19}$$

for all $x \in X\backslash\{0\}$ with $T(0) = f(0)$. In particular, T is given by

$$\begin{aligned} T(x) &= \lim_{n\to\infty} \frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right) + f\left(\frac{-x}{2^n}\right) - 2f(0)\right) \\ &\quad + \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} + f(0) \end{aligned}$$

for all $x \in X$.

Proof. Let $J_n f : X \to Y$ be a mapping defined by

$$\begin{aligned} J_n f(x) &= \lim_{n\to\infty} \frac{4^n}{2}\left(f(2^{-n}x) + f(-2^{-n}x) - 2f(0)\right) \\ &\quad + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} + f(0) \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and

$$\begin{aligned} &\|J_j f(x) - J_{j+1} f(x)\| \\ &= \left\| 2^{2j-1} Df\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right) + 2^{2j-1} Df\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right. \\ &\qquad \left. - \frac{Df(2^j x, -2^j x)}{2^{j+2}} + \frac{Df(-2^j x, 2^j x)}{2^{j+2}} \right\| \\ &\le \max\left\{ |2|^{2j-1}\varphi\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right), |2|^{2j-1}\varphi\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right), \right. \end{aligned}$$

$$\left. \frac{\varphi(2^j x, -2^j x)}{|2|^{j+2}}, \frac{\varphi(-2^j x, 2^j x)}{|2|^{j+2}} \right\} \tag{2.20}$$

for all $x \in X\backslash\{0\}$ and all $j \ge 0$. It follows from (2.15), (2.16) and (2.20) that the sequence $\{J_n f(x)\}$ is Cauchy for all $x \in X\backslash\{0\}$. Since Y is complete and $J_nf(0) = f(0)$ for all $n \in \mathbb{N}$, we conclude that $\{J_nf(x)\}$ is convergent for all $x \in X$. Set

$$T(x) := \lim_{n\to\infty} J_nf(x).$$

From (2.20) we have

$$\begin{aligned}\|J_nf(x) - f(x)\| \leq \quad & \max_{0\leq j<n}\left\{|2|^{2j-1}\varphi\left(\tfrac{x}{2^{j+1}}, \tfrac{-x}{2^{j+1}}\right), |2|^{2j-1}\varphi\left(\tfrac{-x}{2^{j+1}}, \tfrac{x}{2^{j+1}}\right),\right. \\ (2.21) \qquad & \left.\tfrac{\varphi(2^jx,-2^jx)}{|2|^{j+2}}, \tfrac{\varphi(-2^jx,2^jx)}{|2|^{j+2}}\right\}\end{aligned}$$

for all $n \in N$ and all $x \in X\backslash\{0\}$. By taking n to approach infinity in (2.21) and using (2.17) one obtains (2.19). By using (2.18) we get

$$\begin{aligned}\|DJ_nf(x,y)\| &= \left\|\frac{Df(2^nx, 2^ny) - Df(-2^nx, -2^ny)}{2^{n+1}}\right. \\ &\qquad \left. +2^{2n-1}Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 2^{2n-1}Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right\| \\ &\leq \max\left\{\frac{\varphi(2^nx, 2^ny)}{|2|^{n+1}}, \frac{\varphi(-2^nx, -2^ny)}{|2|^{n+1}},\right. \\ &\qquad \left.|2|^{2n-1}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right), |2|^{2n-1}\varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right\}\end{aligned}$$

for all $x, y \in X\backslash\{0\}$ and all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ and using (2.15), (2.16) and Lemma 2.1 we get $DT(x,y) = 0$ for all $x, y \neq 0$ and so T is a general quadratic mapping. Now we are going to prove the uniqueness of T. Assume that T' is another general quadratic mapping satisfying (2.19) with $T'(0) = f(0)$. Then

$$\begin{aligned}T'(x) &= \sum_{j=0}^{k-1}\left(2^{2j-1}DT'\left(\frac{x}{2^{j+1}}, \frac{-x}{2^{j+1}}\right) + 2^{2j-1}DT'\left(\frac{-x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\right. \\ &\qquad \left.-\frac{DT'(2^jx, -2^jx)}{2^{j+2}} + \frac{DT'(-2^jx, 2^jx)}{2^{j+2}}\right) + J_kT'(x) \\ &= J_kT'(x)\end{aligned}$$

for any $k \in \mathbb{N}$ and so

$$
\begin{aligned}
&\|T(x)-T'(x)\| \\
&= \lim_{k\to\infty} \|J_{2k}T(x)-J_{2k}T'(x)\| \\
&\le \lim_{k\to\infty} \max\{\|J_{2k}T(x)-J_{2k}f(x)\|, \|J_{2k}f(x)-J_{2k}T'(x)\|\} \\
&\le \lim_{k\to\infty} \max\Bigg\{ \frac{\|T(2^{2k}x)-f(2^{2k}x)\|}{|2|^{2k+1}}, \frac{\|T(-2^{2k}x)-f(-2^{2k}x)\|}{|2|^{2k+1}}, \\
&\quad \frac{\|f(2^{2k}x)-T'(2^{2k}x)\|}{|2|^{2k+1}}, \frac{\|f(-2^{2k}x)-T'(-2^{2k}x)\|}{|2|^{2k+1}}, \\
&\quad |2|^{4k-1}\left\|T\left(\frac{x}{2^{2k}}\right)-f\left(\frac{x}{2^{2k}}\right)\right\|, |2|^{4k-1}\left\|T\left(-\frac{x}{2^{2k}}\right)-f\left(-\frac{x}{2^{2k}}\right)\right\|, \\
&\quad |2|^{4k-1}\left\|f\left(\frac{x}{2^{2k}}\right)-T'\left(\frac{x}{2^{2k}}\right)\right\|, |2|^{4k-1}\left\|f\left(-\frac{x}{2^{2k}}\right)-T'\left(-\frac{x}{2^{2k}}\right)\right\| \Bigg\} \\
&\le \lim_{k\to\infty}\lim_{n\to\infty}\max_{0\le j<n} \Bigg\{ |2|^{2j-2k-2}\varphi\left(\frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}}\right), \\
&\qquad |2|^{2j-2k-2}\varphi\left(\frac{-2^{2k}x}{2^{j+1}}, \frac{2^{2k}x}{2^{j+1}}\right) \frac{\varphi(2^{2k+j}x, -2^{2k+j}x)}{|2|^{2k+j+3}}, \\
&\qquad \frac{\varphi(-2^{2k+j}x, 2^{2k+j}x)}{|2|^{2k+j+3}}, |2|^{4k+2j-2}\varphi\left(\frac{x}{2^{2k+j+1}}, \frac{-x}{2^{2k+j+1}}\right), \\
&\qquad |2|^{4k+2j-2}\varphi\left(\frac{-x}{2^{2k+j+1}}, \frac{x}{2^{2k+j+1}}\right), \\
&\qquad \frac{\varphi(2^{j-2k}x, -2^{j-2k}x)}{|2|^{j-4k+3}}, \frac{\varphi(-2^{j-2k}x, 2^{j-2k}x)}{|2|^{j-4k+3}} \Bigg\}
\end{aligned} \tag{2.22}
$$

for all $x \in X\backslash\{0\}$ and all $k \in \mathbb{N}$. On the other hand, we have the inequalties

$$
\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\max_{0\le j<n}\left\{ |2|^{2j-2k-2}\varphi\left(\frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}}\right)\right\} \\
&\le \lim_{k\to\infty}\lim_{n\to\infty}\max\Bigg\{ \max_{0\le j<k}\left\{ |2|^{2j-2k-2}\varphi\left(\frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}}\right)\right\}, \\
&\qquad \max_{k\le j<2k}\left\{ |2|^{2j-2k-2}\varphi\left(\frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}}\right)\right\}, \\
&\qquad \max_{2k\le j<n}\left\{ |2|^{2j-2k-2}\varphi\left(\frac{2^{2k}x}{2^{j+1}}, \frac{-2^{2k}x}{2^{j+1}}\right)\right\} \Bigg\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \lim_{k\to\infty} \max\left\{|2|^{-4}\max_{k\leq j<2k}\left\{\frac{\varphi(2^jx,2^jx)}{|2|^j}\right\}, |2|^{k-4}\max_{0\leq j<k}\left\{\frac{\varphi(2^jx,2^jx)}{|2|^j}\right\},\right.\\
&\qquad\qquad \left.|2|^{2k-2}\lim_{n\to\infty}\max_{0\leq j<n-2k}\left\{|4|^j\varphi\left(\frac{x}{2^{j+1}},\frac{-x}{2^{j+1}}\right)\right\}\right\}\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\max_{0\leq j<n}\left\{\frac{\varphi(2^{2k+j}x,-2^{2k+j}x)}{|2|^{2k+j+3}}\right\}\\
&\leq \lim_{k\to\infty}\lim_{n\to\infty}\max_{2k\leq j<n-2k}\left\{\frac{\varphi(2^jx,-2^jx)}{|2|^{j+3}}\right\}\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\max_{0\leq j<n}\left\{|2|^{4k+2j-2}\varphi\left(\frac{-x}{2^{2k+j+1}},\frac{x}{2^{2k+j+1}}\right)\right\}\\
&\leq \lim_{k\to\infty}\lim_{n\to\infty}\max_{2k\leq j<n-2k}\left\{\|2|^{2j-2}\varphi\left(\frac{-x}{2^{j+1}},\frac{x}{2^{j+1}}\right)\right\}\\
&=0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\max_{0\leq j<n}\left\{\frac{\varphi(-2^{j-2k}x,2^{j-2k}x)}{|2|^{j-4k+3}}\right\}\\
&\leq \lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\max_{0\leq j<k}\left\{\frac{\varphi(-2^{j-2k}x,2^{j-2k}x)}{|2|^{j-4k+3}}\right\},\right.\\
&\qquad\qquad \max_{k\leq j<2k}\left\{\frac{\varphi(-2^{j-2k}x,2^{j-2k}x)}{|2|^{j-4k+3}}\right\},\\
&\qquad\qquad \left.\max_{2k\leq j<n}\left\{\frac{\varphi(-2^{j-2k}x,2^{j-2k}x)}{|2|^{j-4k+3}}\right\}\right\}\\
&\leq \lim_{k\to\infty}\max\left\{|2|^{-4}\max_{k+1\leq j<2k+1}\left\{|4|^j\varphi\left(\frac{x}{2^j},\frac{-x}{2^j}\right)\right\},\right.\\
&\qquad\qquad |2|^{k-4}\max_{1\leq j<k+1}\left\{|4|^j\varphi\left(\frac{x}{2^j},\frac{-x}{2^j}\right)\right\},\\
&\qquad\qquad \left.|2|^{2k-3}\lim_{n\to\infty}\max_{0\leq j<n-2k}\left\{\frac{\varphi(2^jx,2^jx)}{|2|^j}\right\}\right\}\\
&=0
\end{aligned}$$

for all $x \in X \backslash \{0\}$ and all $k \in \mathbb{N}$. So the right hand side of (2.22) tends to 0 as $k \to \infty$. Since $T(0) = f(0) = T'(0)$, we conclude that $T(x) = T'(x)$ for all $x \in X$. This completes the proof of the uniqueness of T. $\square$

Corollary 2.7. *Let $1 < r < 2$ be a real number and $|2| < 1$. If $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X \backslash \{0\}$, then there exists a unique general quadratic mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq 2\theta|2|^{-1-r}\|x\|^r \tag{2.23}$$

for all $x \in X \backslash \{0\}$ with $T(0) = f(0)$.

Proof. Let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$. Since $|2| < 1$ and $1 < r < 2$, we have

$$\lim_{n\to\infty} |4|^n \varphi(2^{-n}x, 2^{-n}y) = \lim_{n\to\infty} |2|^{n(2-r)}\varphi(x, y) = 0$$

and

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = \lim_{n\to\infty} |2|^{n(r-1)}\varphi(x, y) = 0$$

for all $x, y \in X$. Therefore the conditions of Theorem 2.6 are fulfilled and it is easy to see that $\tilde{\varphi}(x) = 2\theta|2|^{-1-r}\|x\|^r$. By Theorem 2.6 there is a unique general quadratic mapping $T : X \to Y$ satisfying (2.23) with $T(0) = f(0)$. $\square$

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64-66.

[2] I. S. Chang, E. H. Lee, and H. M. Kim, *On Hyers-Ulam-Rassias stability of a quadratic functional equation*, Math. Inequal. Appl. **6** (2003), 87-95.

[3] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431-436.

[4] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222-224.

[5] S. S. Jin and Y. H. Lee, *A fixed point approach to the stability of the generalized polynomial functional equation of degree 2*, Commun. Korean Math. Soc. **28** (2013), 269-283.

[6] K. W. Jun and Y. H. Lee, *A Generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equations II*, Kyungpook Math. J. **47** (2007), 91-103.

[7] G. H. Kim, *On the stability of functional equations with square-symmetric operation*, Math. Inequal. Appl. **4** (2001), 257-266.

[8] H. M. Kim, *On the stability problem for a mixed type of quartic and quadratic functional equation*, J. Math. Anal. Appl. **324** (2006), 358-372.

[9] Y. H. Lee, *On the Hyers-Ulam-Rassias stability of the generalized polynomial function of degree 2*, J. Chungcheong Math. Soc. **22** (2009), 201-209.
[10] Y. H. Lee, *On the stability of the monomial functional equation*, Bull. Korean Math. Soc. **45** (2008), 397-403.
[11] Y. H. Lee and K. W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238** (1999), 305-315.
[12] Y. H. Lee and K. W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Pexider equation*, J. Math. Anal. Appl. **246** (2000), 627-638.
[13] Y. H. Lee and K. W. Jun, *On the stability of approximately additive mappings*, Proc. Amer. Math. Soc. **128** (2000), 1361-1369.
[14] M. S. Moslehian and Th. M. Rassias , *Stability of functional equations in non-Archimedean spaces*, Appl. Anal. Discrete Math. **1** (2007), 325-334.
[15] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.
[16] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

*
Department of Mathematics Education
Gongju National University of Education
Gongju 314-060, Republic of Korea
E-mail: chjlee@gjue.ac.kr

**
Department of Mathematics Education
Gongju National University of Education
Gongju 314-060, Republic of Korea
E-mail: lyhmzi@gjue.ac.kr