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FURTHER LOG-SINE AND LOG-COSINE INTEGRALS

Junesang Choi*

Abstract. Motivated essentially by their potential for applications in a wide range of mathematical and physical problems, the log-sine and log-cosine integrals have been evaluated, in the existing literature on the subject, in many different ways. Very recently, Choi [6] presented explicit evaluations of some families of log-sine and log-cosine integrals by making use of the familiar Beta function. In the present sequel to the investigation [6], we evaluate the log-sine and log-cosine integrals involved in more complicated integrands than those in [6], by also using the Beta function.

1. Introduction and preliminaries

Motivated essentially by their potential for applications in a wide range of mathematical and physical problems, the log-sine and log-cosine integrals have been evaluated, in the existing literature on the subject, in many different ways (see, for example, [2, 3, 6, 7, 9, 10, 13, 15, 17, 18] and the references therein). By making use of the familiar Beta function $B(\alpha,\,\beta)$ (see, for example, [18, p. 8, Eq. (43)])

$$B(\alpha,\,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (\Re(\alpha)>0;\ \Re(\beta)>0)\\[2ex] \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\Re(\alpha)<0;\ \Re(\beta)<0;\ \alpha,\,\beta\notin\mathbb{Z}_0^-),\end{cases}\tag{1.1}$$

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where $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$, $\mathbb{Z}^- := \{-1, -2, -3, \ldots\}$, and Γ is the familiar Gamma function, Choi [6] presented evaluations of some families of log-sine and log-cosine integrals. In the present sequel to the investigation [6], we evaluate the log-sine and log-cosine integrals involved in more complicated integrands than those in [6], by also using the Beta function (1.1).

Setting $t = \sin^2\theta$ in (1.1), we have the following equivalent form of the Beta function $B(\alpha, \beta)$:

(1.2)
$$B(\alpha, \beta) = 2\int_0^{\pi/2} (\sin\theta)^{2\alpha-1}\,(\cos\theta)^{2\beta-1}\,d\theta \quad (\Re(\alpha) > 0;\ \Re(\beta) > 0).$$

For our purpose, let us replace α and β by $\mu+1$ and $\nu+1$, respectively, in (1.2) and we get:

$$\int_0^{\pi/2} (\sin\theta)^{2\mu+1}\,(\cos\theta)^{2\nu+1}\,d\theta = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{2\,\Gamma(\mu+\nu+2)} \tag{1.3}$$
$$(\Re(\mu) > -1;\ \Re(\nu) > -1).$$

Differentiating each side of (1.3) p times with respect to the variable μ, we obtain

$$\int_0^{\pi/2} (\log\,\sin\theta)^p\,(\sin\theta)^{2\mu+1}\,(\cos\theta)^{2\nu+1}\,d\theta$$
$$= \frac{1}{2^{p+1}}\,\frac{\partial^p}{\partial\mu^p}\left[\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}\right] \tag{1.4}$$
$$(\Re(\mu) > -1;\ \Re(\nu) > -1;\ p \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}),$$

where $\mathbb{N}$ is the set of positive integers. Differentiating each side of (1.4) q times with respect to the variable ν, we get

$$\int_0^{\pi/2} (\log\,\sin\theta)^p\,(\log\,\cos\theta)^q\,(\sin\theta)^{2\mu+1}\,(\cos\theta)^{2\nu+1}\,d\theta$$
$$= \frac{1}{2^{p+q+1}}\,\frac{\partial^q}{\partial\nu^q}\,\frac{\partial^p}{\partial\mu^p}\left[\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}\right] \tag{1.5}$$
$$(\Re(\mu) > -1;\ \Re(\nu) > -1;\ p, q \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}).$$

We also need the following notations and functions. The generalized harmonic numbers $H_n^{(s)}$ of order s are defined by (*cf.* [1]; see also [11], [17, p. 156] and [18, Section 3.5])

$$H_n^{(s)} := \sum_{j=1}^{n} \frac{1}{j^s} \quad (n \in \mathbb{N};\ s \in \mathbb{C}) \tag{1.6}$$

and

$$H_n := H_n^{(1)} = \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}) \tag{1.7}$$

are the harmonic numbers. Here $\mathbb{C}$ denotes the set of complex numbers, and we assume that

$$H_0 := 0, \quad H_0^{(s)} := 0 \quad (s \in \mathbb{C} \setminus \{0\}) \quad \text{and} \quad H_0^{(0)} := 1.$$

The generalized harmonic functions $H_n^{(s)}(z)$ are defined by (see [4, 11]; see also [14, 16])

$$H_n^{(s)}(z) := \sum_{j=1}^{n} \frac{1}{(j+z)^s} \quad (n \in \mathbb{N};\ z \in \mathbb{C} \setminus \mathbb{Z}^-;\ s \in \mathbb{C}), \tag{1.8}$$

so that, obviously,

$$H_n^{(s)}(0) = H_n^{(s)}.$$

The generalized odd harmonic numbers $O_n^{(s)}$ of order s are defined by

$$O_n^{(s)} := \sum_{j=1}^{n} \frac{1}{(2j-1)^s} \quad (n \in \mathbb{N};\ s \in \mathbb{C}) \tag{1.9}$$

and

$$O_n := O_n^{(1)} = \sum_{j=1}^{n} \frac{1}{2j-1} \quad (n \in \mathbb{N}) \tag{1.10}$$

are the odd harmonic numbers.

The Riemann Zeta function $\zeta(s)$ is defined by (see, for example, [18, Section 2.3])

(1.11)

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1), \end{cases}$$

which is an obvious special case of the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \quad \left(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{1.12}$$

Equation (1.6) can be written in the following form:

$$H_n^{(s)} = \zeta(s) - \zeta(s, n+1) \qquad (\Re(s) > 1;\ n \in \mathbb{N}) \tag{1.13}$$

by recalling the well-known (easily-derivable) relationship between the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ (see [17, Eq. 2.3(9)]):

$$\zeta(s) = \zeta(s, n+1) + \sum_{k=1}^{n} k^{-s} \quad (n \in \mathbb{N}_0). \tag{1.14}$$

The Polygamma functions $\psi^{(n)}(s)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(s) := \frac{d^{n+1}}{dz^{n+1}} \log\Gamma(s) = \frac{d^n}{ds^n}\psi(s) \quad \left(n \in \mathbb{N}_0;\ s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \tag{1.15}$$

where $\psi(s)$ denotes the Psi (or Digamma) function defined by

$$\psi(s) := \frac{d}{ds} \log\Gamma(s) \quad \text{and} \quad \psi^{(0)}(s) = \psi(s) \quad \left(s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{1.16}$$

A well-known (and potentially useful) relationship between the Polygamma functions $\psi^{(n)}(s)$ and the generalized Zeta function $\zeta(s,a)$ is given by

$$\psi^{(n)}(s) = (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(k+s)^{n+1}} = (-1)^{n+1}\, n!\, \zeta(n+1, s) \tag{1.17}$$

$$\left(n \in \mathbb{N};\ s \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

It is also easy to have the following expression (*cf.* [17, Eq. 1.2(54)]):

$$\psi^{(m)}(s+n) - \psi^{(m)}(s) = (-1)^m\, m!\, H_n^{(m+1)}(s-1) \quad (m,\, n \in \mathbb{N}_0), \tag{1.18}$$

which immediately gives $H_n^{(s)}$ the following another expression (*cf.* [12, Eq. (20)]):

$$H_n^{(m)} = \frac{(-1)^{m-1}}{(m-1)!} \left[\psi^{(m-1)}(n+1) - \psi^{(m-1)}(1)\right] \quad (m \in \mathbb{N};\ n \in \mathbb{N}_0). \tag{1.19}$$

2. Some properties and special values of the Psi and Gamma functions

In this section, for easy reference in the next section, we recall some properties and special values of the Psi and Gamma functions. The fundamental functional relation for the Gamma function is given as follows:

$$\Gamma(z+1) = z\,\Gamma(z), \tag{2.1}$$

so that, obviously,

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \quad (n \in \mathbb{N}_0)\,. \tag{2.2}$$

$$\psi(z+n) = \psi(z) + H_n(z-1) \quad (n \in \mathbb{N}_0)\,, \tag{2.3}$$

which is an obvious special case $(m = 0)$ of (1.18).

$$\Gamma(n+1) = n! \quad \text{and} \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{2^{2n}\,n!} \quad (n \in \mathbb{N}_0)\,. \tag{2.4}$$

$$\psi(n) = -\gamma + H_{n-1} \quad (n \in \mathbb{N})\,, \tag{2.5}$$

where $H_0 := 0$ and γ denotes the Euler-Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log\,n\right) \cong 0.57721\,56649\,01532\,86060\,6512\cdots\,. \tag{2.6}$$

$$\psi\left(n+\frac{1}{2}\right) = -\gamma - 2\,\log\,2 + 2\,O_n \quad (n \in \mathbb{N}_0)\,. \tag{2.7}$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2. \tag{2.8}$$

$$H_n^{(s)}\left(-\frac{1}{2}\right) = 2^s\,O_n^{(s)} \quad (n \in \mathbb{N};\ s \in \mathbb{C}). \tag{2.9}$$

$$\begin{aligned}\psi'(n+1) - \psi'\left(n+m+\frac{5}{2}\right)\\ = -\frac{\pi^2}{3} + 4\,O_{n+m+2}^{(2)} - H_n^{(2)} \quad (n,\,m \in \mathbb{N}_0)\,.\end{aligned} \tag{2.10}$$

$$\psi^{(2)}(n+1)-\psi^{(2)}\left(n+m+\frac{5}{2}\right) = 2\left(6\,\zeta(3)+H_n^{(3)}+8\,O_{n+m+2}^{(3)}\right) \quad (n,\, m\in\mathbb{N}_0)\,. \tag{2.11}$$

$$\psi^{(2)}\left(n+\frac{3}{2}\right)-\psi^{(2)}\left(n+m+\frac{5}{2}\right) = -2\,H_{m+1}^{(3)}\left(n+\frac{1}{2}\right) \quad (n,\, m\in\mathbb{N}_0)\,. \tag{2.12}$$

3. Log-Sine and its related integrals

For convenience, let

$$f(\mu,\nu) := \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}.$$

Taking the logarithmic partial derivative of $f(\mu,\nu)$ with respect to the variable μ, we get

$$\frac{\partial}{\partial\mu}f(\mu,\nu) = f(\mu,\nu)\,[\psi(\mu+1)-\psi(\mu+\nu+2)]\,. \tag{3.1}$$

Then, applying Leibniz's rule for differentiation to Equation (3.1), we obtain

$$\frac{\partial^{k+1}}{\partial\mu^{k+1}}f(\mu,\nu) = \sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j}{\partial\mu^j}f(\mu,\nu) \cdot \frac{\partial^{k-j}}{\partial\mu^{k-j}}\left[\psi(\mu+1)-\psi(\mu+\nu+2)\right] \quad (k\in\mathbb{N}_0)\,. \tag{3.2}$$

Now, from Equations (1.4), (3.1) and (3.2), we find the following formula:

$$\int_0^{\pi/2}(\log\,\sin\theta)^2\,(\sin\theta)^{2\mu+1}\,(\cos\theta)^{2\nu+1}\,d\theta = \frac{1}{8}\,\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)} \cdot\left[(\psi(\mu+1)-\psi(\mu+\nu+2))^2+\psi'(\mu+1)-\psi'(\mu+\nu+2)\right] \quad (\Re(\mu)>-1;\ \Re(\nu)>-1)\,, \tag{3.3}$$

where the prime$'$ indicates the differentiation with respect to the variable μ.

Here, we present some interesting special cases of (3.3) without their proofs. It is noted that each of the formulas presented here can be easily derived by using suitably chosen identities and, if necessary, their easily-derivable variants, given in Sections 1 and 2.

(1) $(\mu,\,\nu) = (n,\,m) \in \mathbb{N}_0 \times \mathbb{N}_0$.

$$\begin{aligned}&\int_0^{\pi/2} (\log\,\sin\theta)^2\,(\sin\theta)^{2n+1}\,(\cos\theta)^{2m+1}\,d\theta\\ &= \frac{n!\,m!}{8\,(n+m+1)!}\,\left[(H_n - H_{n+m+1})^2 + H_{m+1}^{(2)}(n)\right].\end{aligned} \tag{3.4}$$

(2) $(\mu,\,\nu) = (n,\,m+1/2)$ and $(n,\,m) \in \mathbb{N}_0 \times \mathbb{N}_0$.

$$\begin{aligned}&\int_0^{\pi/2} (\log\,\sin\theta)^2\,(\sin\theta)^{2n+1}\,(\cos\theta)^{2(m+1)}\,d\theta\\ &= \frac{2^{2n-1}\,n!\,(2m+2)!\,(n+m+2)!}{(m+1)!\,(2n+2m+4)!}\\ &\quad\cdot\left[(2\,\log 2 + H_n - 2\,O_{n+m+2})^2 - \frac{\pi^2}{3} + 4\,O_{n+m+2}^{(2)} - H_n^{(2)}\right].\end{aligned} \tag{3.5}$$

(3) $(\mu,\,\nu) = (n+1/2,\,m)$ and $(n,\,m) \in \mathbb{N}_0 \times \mathbb{N}_0$.

$$\begin{aligned}&\int_0^{\pi/2} (\log\,\sin\theta)^2\,(\sin\theta)^{2(n+1)}\,(\cos\theta)^{2m+1}\,d\theta\\ &= \frac{2^{2m-1}\,m!\,(2n+2)!\,(m+n+2)!}{(n+1)!\,(2m+2n+4)!}\\ &\quad\cdot\left[4\,(O_{n+1} - O_{n+m+2})^2 + H_{m+1}^{(2)}\left(n+\frac{1}{2}\right)\right].\end{aligned} \tag{3.6}$$

(4) $(\mu,\,\nu) = (n+1/2,\,m+1/2)$ and $(n,\,m) \in \mathbb{N}_0 \times \mathbb{N}_0$.

$$\begin{aligned}&\int_0^{\pi/2} (\log\,\sin\theta)^2\,(\sin\theta)^{2(n+1)}\,(\cos\theta)^{2(m+1)}\,d\theta\\ &= \frac{\pi\,(2n+2)!\,(2m+2)!}{2^{2n+2m+7}\,(n+1)!\,(m+1)!\,(n+m+2)!}\\ &\quad\cdot\left[(2\,\log 2 - 2\,O_{n+1} + H_{n+m+2})^2 + \frac{\pi^2}{3} - 4\,O_{n+1}^{(2)} + H_{n+m+2}^{(2)}\right].\end{aligned} \tag{3.7}$$

Differentiating both sides of Equation (3.3) with respect to μ, we obtain

(3.8)
$$\begin{aligned}&\int_0^{\pi/2} (\log\,\sin\theta)^3\,(\sin\theta)^{2\mu+1}\,(\cos\theta)^{2\nu+1}\,d\theta\\&=\frac{1}{16}\,\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}\,\big[(\psi(\mu+1)-\psi(\mu+\nu+2))^3\\&\quad+3(\psi(\mu+1)-\psi(\mu+\nu+2))\,(\psi'(\mu+1)-\psi'(\mu+\nu+2))\\&\quad+\psi^{(2)}(\mu+1)-\psi^{(2)}(\mu+\nu+2)\big]\quad(\Re(\mu)>-1;\ \Re(\nu)>-1)\,.\end{aligned}$$

Here we give four special cases of Equation (3.8).

(1) $(\mu,\,\nu)=(n,\,m)\in\mathbb{N}_0\times\mathbb{N}_0$.

(3.9)
$$\begin{aligned}&\int_0^{\pi/2} (\log\,\sin\theta)^3\,(\sin\theta)^{2n+1}\,(\cos\theta)^{2m+1}\,d\theta\\&=\frac{n!\,m!}{16\,(n+m+1)!}\\&\quad\cdot\Big[(H_n-H_{n+m+1})^3+3\,(H_n-H_{n+m+1})\;H^{(2)}_{m+1}(n)-2\,H^{(3)}_{m+1}(n)\Big]\,.\end{aligned}$$

(2) $(\mu,\,\nu)=(n,\,m+1/2)$ and $(n,\,m)\in\mathbb{N}_0\times\mathbb{N}_0$.

(3.10)
$$\begin{aligned}&\int_0^{\pi/2} (\log\,\sin\theta)^3\,(\sin\theta)^{2n+1}\,(\cos\theta)^{2(m+1)}\,d\theta\\&=\frac{2^{2n-2}\,n!\,(2m+2)!\,(n+m+2)!}{(m+1)!\,(2n+2m+4)!}\,\Big[(2\,\log 2+H_n-2\,O_{n+m+2})^3\\&\quad+3\;(2\,\log 2+H_n-2\,O_{n+m+2})\,\left(-\frac{\pi^2}{3}+4\,O^{(2)}_{n+m+2}-H^{(2)}_n\right)\\&\quad+2\left(6\,\zeta(3)+H^{(3)}_n+8\,O^{(3)}_{n+m+2}\right)\Big]\,.\end{aligned}$$

(3) $(\mu,\,\nu)=(n+1/2,\,m)$ and $(n,\,m)\in\mathbb{N}_0\times\mathbb{N}_0$.

(3.11)
$$\begin{aligned}&\int_0^{\pi/2} (\log\,\sin\theta)^3\,(\sin\theta)^{2(n+1)}\,(\cos\theta)^{2m+1}\,d\theta\\&=\frac{2^{2m-2}\,m!\,(2n+2)!\,(m+n+2)!}{(n+1)!\,(2m+2n+4)!}\Big[8\,(O_{n+1}-O_{n+m+2})^3\\&\quad+6\;(O_{n+1}-O_{n+m+2})\;H^{(2)}_{m+1}\left(n+\frac{1}{2}\right)-2\,H^{(3)}_{m+1}\left(n+\frac{1}{2}\right)\Big]\,.\end{aligned}$$

(4) $(\mu,\,\nu) = (n+1/2,\,m+1/2)$ and $(n,\,m) \in \mathbb{N}_0 \times \mathbb{N}_0$.

(3.12)
$$\begin{aligned}
&\int_0^{\pi/2} (\log\,\sin\theta)^3\,(\sin\theta)^{2(n+1)}\,(\cos\theta)^{2(m+1)}\,d\theta \\
&\quad = \frac{\pi\,(2n+2)!\,(2m+2)!}{2^{2n+2m+8}\,(n+1)!\,(m+1)!\,(n+m+2)!} \\
&\qquad \cdot \Big[\,(-2\,\log 2 + 2\,O_{n+1} - H_{n+m+2})^3 \\
&\qquad\quad + 3\,(-2\,\log 2 + 2\,O_{n+1} - H_{n+m+2})\left(\frac{\pi^2}{3} - 4\,O_{n+1}^{(2)} + H_{n+m+2}^{(2)}\right) \\
&\qquad\quad + 2\left(-6\,\zeta(3) - H_{n+m+2}^{(3)} + 8\,O_{n+1}^{(3)}\right)\Big].
\end{aligned}$$

The case of (1.5) when $p = 1 = q$ gives

$$\begin{aligned}
&\int_0^{\pi/2} (\log\,\sin\theta)\,(\log\,\cos\theta)\,(\sin\theta)^{2\mu+1}\,(\cos\theta)^{2\nu+1}\,d\theta \\
&= \frac{1}{8}\,\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)} \cdot [(\psi(\mu+1) - \psi(\mu+\nu+2))(\psi(\nu+1) \\
&\qquad\qquad - \psi(\mu+\nu+2)) - \psi'(\mu+\nu+2)] \\
&\qquad\qquad (\Re(\mu) > -1;\ \Re(\nu) > -1)\,.
\end{aligned} \tag{3.13}$$

Here we give four special cases of Equation (3.13).

(1) $(\mu,\,\nu) = (n,\,m) \in \mathbb{N}_0 \times \mathbb{N}_0$.

(3.14)
$$\begin{aligned}
&\int_0^{\pi/2} (\log\,\sin\theta)\,(\log\,\cos\theta)\,(\sin\theta)^{2n+1}\,(\cos\theta)^{2m+1}\,d\theta \\
&= \frac{n!\,m!}{8\,(n+m+1)!}\,\left[(H_n - H_{n+m+1})\,(H_m - H_{n+m+1}) + H_{n+m+1}^{(2)} - \frac{\pi^2}{6}\right].
\end{aligned}$$

(2) $(\mu,\,\nu) = (n,\,m+1/2)$ and $(n,\,m) \in \mathbb{N}_0 \times \mathbb{N}_0$.

(3.15)
$$\begin{aligned}
&\int_0^{\pi/2} (\log\,\sin\theta)\,(\log\,\cos\theta)\,(\sin\theta)^{2n+1}\,(\cos\theta)^{2(m+1)}\,d\theta \\
&= \frac{2^{2n-1}\,n!\,(2m+2)!\,(n+m+2)!}{(m+1)!\,(2n+2m+4)!} \\
&\qquad \cdot \left[2\,(O_{m+1} - O_{n+m+2})\,(2\log 2 + H_n - 2\,O_{n+m+2}) + 4\,O_{n+m+2}^{(2)} - \frac{\pi^2}{2}\right].
\end{aligned}$$

(3) $(\mu,\,\nu)=(n+1/2,\,m)$ and $(n,\,m)\in\mathbb{N}_0\times\mathbb{N}_0$.

(3.16)
$$\int_0^{\pi/2}(\log\,\sin\theta)\,(\log\,\cos\theta)\,(\sin\theta)^{2(n+1)}\,(\cos\theta)^{2m+1}\,d\theta$$
$$=\frac{2^{2m-1}\,m!\,(2n+2)!\,(n+m+2)!}{(n+1)!\,(2n+2m+4)!}$$
$$\cdot\left[2\,(O_{n+1}-O_{n+m+2})\,(2\log 2+H_m-2\,O_{n+m+2})+4\,O^{(2)}_{n+m+2}-\frac{\pi^2}{2}\right].$$

(4) $(\mu,\,\nu)=(n+1/2,\,m+1/2)$ and $(n,\,m)\in\mathbb{N}_0\times\mathbb{N}_0$.

(3.17)
$$\int_0^{\pi/2}(\log\,\sin\theta)\,(\log\,\cos\theta)\,(\sin\theta)^{2(n+1)}\,(\cos\theta)^{2(m+1)}\,d\theta$$
$$=\frac{\pi\,(2n+2)!\,(2m+2)!}{2^{2n+2m+7}\,(n+1)!\,(m+1)!\,(n+m+2)!}\Big[H^{(2)}_{n+m+2}-\frac{\pi^2}{6}$$
$$+\,(-2\log 2+2\,O_{n+1}-H_{n+m+2})\,(-2\log 2+2\,O_{m+1}-H_{n+m+2})\Big].$$

4. Concluding remarks

In view of Equation (1.1), it is easy to see the following symmetric relation with respect to the variables α and β:

$$B(\alpha,\beta)=B(\beta,\alpha). \tag{4.1}$$

Since we begin by giving a partial differentiation to the Equation (1.3), which preserves the symmetry with respect to the variables μ and ν, if we interchange $\sin\theta\leftrightarrow\cos\theta$ and $\mu\leftrightarrow\nu$ in each of those integrals given in Section 3, we can obtain formulas each of whose evaluations is the same as in the corresponding results in Section 3. For example, we present only one formula corresponding to Equation (3.4):

(4.2)
$$\int_0^{\pi/2}(\log\,\cos\theta)^2\,(\sin\theta)^{2n+1}\,(\cos\theta)^{2m+1}\,d\theta$$
$$=\frac{n!\,m!}{8\,(n+m+1)!}\,\Big[(H_n-H_{n+m+1})^2+H^{(2)}_{m+1}(n)\Big]\quad(n,\,m\in\mathbb{N}_0)\,.$$

Here we showed how we can obtain some interesting and (potentially) useful integrals by using a well known function. We leave the integrals involved in more complicated integrands to the interested reader.

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Department of Mathematics
Dongguk University
Gyeongju 780-714, Republic of Korea
E-mail: junesang@mail.dongguk.ac.kr