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MOMENTS OF LOWER GENERALIZED ORDER STATISTICS FROM DOUBLY TRUNCATED CONTINUOUS DISTRIBUTIONS AND CHARACTERIZATIONS

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Abstract. In this paper, we derive recurrence relations for moments of lower generalized order statistics within a class of doubly truncated distributions. Inverse Weibull, exponentiated Weibull, power function, exponentiated Pareto, exponentiated gamma, generalized exponential, exponentiated log-logistic, generalized inverse Weibull, extended type I generalized logistic, logistic and Gumble distributions are given as illustrative examples. Further, recurrence relations for moments of order statistics and lower record values are obtained as special cases of the lower generalized order statistics, also two theorems for characterizing the general form of distribution based on single moments of lower generalized order statistics are given.

1. Introduction

Kamps [11] introduced the concept of generalized order statistics (*gos*). It is know that ordinary order statistics, upper record values and sequential order statistics are special cases of *gos*. In this paper we will consider the lower generalized order statistics (*lgos*). It can be shown that order statistics, lower record values are special cases of *lgos*.

Let $n \in N$, $k \geq 1$, $m \in \Re$, be the parameters such that

$$\gamma_r = k + (n-r)(m+1) > 0,$$

for all $1 \leq r \leq n$. Then $X^*(1,n,m,k), \cdots, X^*(n,n,m,k)$ are n *lgos* from an absolutely continuous distribution function $(df)F(x)$ with the

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corresponding probability density function (*pdf*) $f(x)$. Their joint *pdf* is

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}[F(x_i)]^{m_i}f(x_i)\Big)[F(x_n)]^{k-1}f(x_n) \tag{1.1}$$

on the cone

$$F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0).$$

For simplicity we shall assume $m_1 = m_2 = \cdots = m_{n-1} = m$.
The marginal *pdf* of the $r-$th $lgos$, $X^*(r,n,m,k)$ is

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint *pdf* of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$, $1 \leq r < s \leq n$ is expressed from (1.1) as

(1.3)

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x)g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y),\ x > y,$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i\ ,\ \ \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}x^{m+1},\ m\neq-1 \\ -lnx,\ m=-1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1),\ x \in [0,1).$$

We shall also take $X^*(0,n,m,k) = 0$. If $m = 0$, $k = 1$, then $X^*(r,n,m,k)$ reduced to the $(n-r+1)-$th order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when $m = -1$, then $X^*(r,n,m,k)$ reduced to the $r-$ th lower k record value [Pawlas and Szynal, [17]]. The work of Burkschat *et al*. [5] may also refer for lower generalized order statistics. Recurrence relations are interesting in their own right. They are useful in reducing the number of operations necessary to obtain a general form for the function under consideration. Furthermore, they are used in characterizing distributions, which in important area, permitting the identification of population distribution from the properties of the sample.

Many recurrence relations between moments of generalized order statistics are available in the literature. Reference may be made to Cramer and Kamps [6], Pawlas and Szynal [17], Ahmad and Fawzy [4], Ahmad [2], Khan *et al*. [16] and references therein.

Characterizations of particular distributions based on the moments and conditional moments of order statistics were presented by some authors such as Wu and Ouyang [18], Grudzien and Szynal [9], Khan and Abouammoh [14], Ahmad [1], Govindarajulu [8], amongs others.

Characterizations based on *gos* have been studied by some authors. Keseling [13] characterized some continuous distributions based on conditional distributions of *gos*. Bieniek and Szynal [4] characterized some distributions via linearity of regression of *gos*. Cramer *et al*. [7] gave a unifying approach on characterization via linear regression of ordered random variables. Khan *et al*. [15] characterized some continuous distributions through conditional expectation of functions of *gos*.

Kamps [12] investigated the importance of recurrence relations of order statistics in characterization.

The doubly truncated case of a distribution is the most general case since it includes the right, left and non-truncated distribution as special cases.

Now if for given P_1 and Q_1

$$\int_{-\infty}^{Q_1} f_1(x)dx = Q \ and \ \int_{-\infty}^{P_1} f_1(x)dx = P, \tag{1.4}$$

where $f_1(x)$ is the *pdf* of X. Then the truncated *pdf* is given by

$$f(x) = \frac{f_1(x)}{P-Q}, \ x \in (Q_1, P_1)$$

and the corresponding *df* by

$$F(x) = \frac{1}{P-Q}[F_1(x) - Q], \ x \in (Q_1, P_1).$$

Suppose the distribution function is of the following general form

$$F_1(x) = e^{-a/h(x)}, \ x \in (\alpha, \ \beta), \tag{1.5}$$

where $a \neq 0$ is a constant and $h(x)$ is continuous, monotonic and differentiable function of x in the interval $[\alpha, \ \beta]$. Then truncated *pdf* $f(x)$ is given by

$$f(x) = \frac{a}{(P-Q)h^2(x)} e^{-a/h(x)} h'(x), \ x \in (Q_1, P_1) \tag{1.6}$$

and the corresponding truncated *pdf* $f(x)$ by

$$F(x) = -Q_2 + \frac{h^2(x)}{ah'(x)} f(x), \tag{1.7}$$

where

$$Q_2 = \frac{Q}{P-Q}.$$

2. Recurrence relations for single moments

THEOREM 2.1. *For the distribution given in (1.7), $n \in N,\ 2 \le r \le n,\ k \ge 1$, and $m > -1$*

(2.1)

$$E[X^{*j}(r,n,m,k)] - E[X^{*j}(r-1,n,m,k)]$$
$$= \frac{j}{a\gamma_r}\Big\{QE[\phi(X^*(r,n,m,k))e^{a/h(X^*(r,n,m,k))}] - E[\phi(X^*(r,n,m,k))]\Big\}, \quad m \ne -1$$

and for $m = -1$

(2.2)

$$E[X^{*j}(r,n,-1,k)] - E[X^{*j}(r-1,n,-1,k)]$$
$$= \frac{j}{ak}\Big\{QE[\phi(X^*(r,n,-1,k))e^{a/h(X^*(r,n,-1,k))}] - E[\phi(X^*(r,n,-1,k))]\Big\}.$$

where

$$\phi(x) = \frac{x^{j-1}h^2(x)}{h'(x)}.$$

Proof. We have from (1.2)

(2.3)

$$E[X^{*j}(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx.$$

Integrating by parts treating $[F(x)]^{\gamma_r-1}f(x)$ for integration and rest of the integrand for differentiation, we get

(2.4)

$$E[X^{*j}(r,n,m,k)]$$
$$= E[X^{*j}(r-1,n,m,k)] - \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_{Q_1}^{P_1} x^{j-1}[F(x)]^{\gamma_r} g_m^{r-1}(F(x))dx$$

the constant of integration vanishes since the integral considered in (2.3) is a definite integral. On using (1.7), we obtain when $m > -1$ that

$$E[X^{*j}(r,n,m,k)] = E[X^{*j}(r-1,n,m,k)] - \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_{Q_1}^{P_1} x^{j-1}[F(x)]^{\gamma_r-1}\Big\{-Q_2+\frac{h^2(x)f(x)}{ah'(x)}\Big\}g_m^{r-1}(F(x))dx$$

which can be written as

$$\begin{aligned}
&E[X^{*j}(r,n,m,k)] - E[X^{*j}(r-1,n,m,k)]\\
&= \frac{jQ_2C_{r-1}}{\gamma_r(r-1)!}\int_{Q_1}^{P_1} x^{j-1}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))dx\\
&\quad - \frac{jC_{r-1}}{a\gamma_r(r-1)!}\int_{Q_1}^{P_1} \frac{x^{j-1}h^2(x)}{h'(x)}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx\\
&= \frac{jQC_{r-1}}{a\gamma_r(r-1)!}\int_{Q_1}^{P_1} \frac{x^{j-1}h^2(x)e^{a/h(x)}}{h'(x)}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx\\
&\quad - \frac{jC_{r-1}}{a\gamma_r(r-1)!}\int_{Q_1}^{P_1} \frac{x^{j-1}h^2(x)}{h'(x)}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx
\end{aligned}$$

and hence the result given in (2.1).
When $m = -1$,

$$\begin{aligned}
&E[X^{*j}(r,n,-1,k)] - E[X^{*j}(r-1,n,-1,k)]\\
&= -\frac{jk^{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}[F(x)]^{k-1}\Big\{-Q_2+\frac{h^2(x)f(x)}{ah'(x)}\Big\}g_{-1}^{r-1}(F(x))dx\\
&= \frac{jQ_2k^{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{j-1}[F(x)]^{k-1}g_{-1}^{r-1}(F(x))dx\\
&\quad - \frac{jk^{r-1}}{a(r-1)!}\int_{Q_1}^{P_1} \frac{x^{j-1}h^2(x)}{h'(x)}[F(x)]^{k-1}f(x)g_{-1}^{r-1}(F(x))dx
\end{aligned}$$

rewriting the above equation we get the result given in (2.2).

Special cases

i) Putting $m = 0,\ k = 1$ in (2.1), we can get the recurrence relations for single moment of order statistics as

(2.5)
$$E(X^j_{n-r+1:n}) = E(X^j_{n-r+2:n}) + \frac{j}{a(n-r+1)}\Big\{QE\Big(\phi(X_{n-r+1:n})e^{a/h(x_{n-r+1:n})}\Big) - E(\phi(X_{n-r+1:n}))\Big\}.$$

That is

$$E(X^j_{r:n}) = E(X^j_{r-1:n}) - \frac{j}{a(r-1)}\Big\{QE\Big(\phi(X_{r-1:n})e^{a/h(x_{r-1:n})}\Big) - E(\phi(X_{r-1:n}))\Big\}.$$

ii) Setting $k = 1$ in (2.2) relations for lower records can be obtained as
(2.6)
$$E(X^j_{L(r)}) = E(X^j_{L(r-1)}) - \frac{j}{a}\Big\{QE\Big(\phi(X_{L(r)})e^{a/h(x_{L(r)})}\Big) - E(\phi(X_{L(r)}))\Big\}.$$

□

Remark 2.2. At $Q = 0$ and $P = 1$, (non-truncated case) relations (2.1) and (2.2) reduce, respectively, to

$$E[X^{*j}(r,n,m,k)] - E[X^{*j}(r-1,n,m,k)] = -\frac{j}{a\gamma_r}E[\phi(X^*(r,n,m,k))]$$

and

$$E[X^{*j}(r,n,-1,k)] - E[X^{*j}(r-1,n,-1,k)] = -\frac{j}{ak}E[\phi(X^*(r,n,-1,k))].$$

The order statistics and lower record values cases are given from (2.5) and (2.6) as

$$E(X^j_{n-r+1:n}) = E(X^j_{n-r+2:n}) - \frac{j}{a(n-r+1)}E(\phi(X_{n-r+1:n})).$$

That is

$$E(X^j_{r:n}) = E(X^j_{r-1:n})] + \frac{j}{a(r-1)}E(\phi(X_{r-1:n}))$$

and

$$E(X^j_{L(r)}) = E(X^j_{L(r-1)}) - \frac{j}{a}E(\phi(X_{L(r)})).$$

Similarly several recurrence relations based on Theorem 2.1 can be established with proper choice of a and $h(x)$.

TABLE 1. Examples Based on Theorem 2.1

Distribution	$\boldsymbol{F(x)}$	$\boldsymbol{a}$	$\boldsymbol{h(x)}$
Inverse Weibull	$e^{-(\theta/x)^p}$ $0 < x < \infty$	θ^p	x^p
Exponentiated Weibull	$[1-e^{-(\lambda x)^p}]^\tau$ $0 < x < \infty$	1	$[-ln\{1-e^{-(\lambda x)^p}\}^\tau]^{-1}$
Power function	$(x/\lambda)^p$ $0 < x < \lambda$	1	$[-ln(x/\lambda)^p]^{-1}$
Exponentiated Pareto	$[1-(1+x)^{-\lambda}]^\theta$ $0 < x < \infty$	1	$[-ln\{1-(1+x)^{-\lambda}\}^\theta]^{-1}$
Exponentiated gamma	$[1-e^{-x}(x+1)]^\theta$ $0 < x < \infty$	1	$[-ln\{1-e^{-x}(x+1)\}^\theta]^{-1}$
Generalized exponential	$[1-e^{-\lambda x}]^\theta$ $0 < x < \infty$	1	$[-ln(1-e^{-\lambda x})^\theta]^{-1}$
Exponentiated log-logistic	$\left[\frac{(x/\sigma)^\beta}{1+(x/\sigma)^\beta}\right]^\theta$ $0 < x < \infty$	1	$\left[-ln\left(\frac{(x/\sigma)^\beta}{1+(x/\sigma)^\beta}\right)^\theta\right]^{-1}$
Generalized inverse Weibull	$e^{-\theta(\alpha/x)^\beta}$ $0 < x < \infty$	θ	$(\alpha/x)^{-\beta}$
Extended type I generalized logistic	$\left(\frac{\lambda}{\lambda+e^{-x}}\right)^p$ $-\infty < x < \infty$	1	$\left[-ln\left(\frac{\lambda}{\lambda+e^{-x}}\right)^p\right]^{-1}$
Logistic	$[1+e^{-x}]^{-1}$ $-\infty < x < \infty$	1	$[ln(1+e^{-x})]^{-1}$
Gumbel	$e^{-e^{-x}}$ $-\infty < x < \infty$	1	e^x

3. Characterization

THEOREM 3.1. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0,\ m > -1$ then*

(3.1)
$$E[X^{*j}(r,n,m,k)] - E[X^{*j}(r-1,n,m,k)]$$
$$= \frac{j}{a\gamma_r}\Big\{QE[\phi(X^*(r,n,m,k))e^{a/h(X^*(r,n,m,k))}] - E[\phi(X^*(r,n,m,k))]\Big\}$$

if and only if

$$F(x) = -Q_2 + \frac{h^2(x)}{ah'(x)}f(x),\ P_1 \le x \le Q_1.$$

Proof. The necessary part follows immediately from equation (2.1). On the other hand if the recurrence relation in equation (3.1) is satisfied, then on using equations (1.2) and (2.4), we have

(3.2)

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$
$$=\frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_{Q_1}^{P_1} x^j[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx$$
$$+\frac{jQC_{r-1}}{a\gamma_r(r-1)!}\int_{Q_1}^{P_1}\frac{x^{j-1}h^2(x)e^{a/h(x)}}{h'(x)}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$
$$-\frac{jC_{r-1}}{a\gamma_r(r-1)!}\int_{Q_1}^{P_1}\frac{x^{j-1}h^2(x)}{h'(x)}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx.$$

Integrating the first integral on the right hand side of equation (3.2), by parts, we get

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$
$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_{Q_1}^{P_1} x^{j-1}[F(x)]^{\gamma_r}g_m^{r-1}(F(x))dx$$
$$+\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^j[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$
$$+\frac{jQC_{r-1}}{a\gamma_r(r-1)!}\int_{Q_1}^{P_1}\frac{x^{j-1}h^2(x)e^{a/h(x)}}{h'(x)}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$
$$-\frac{jC_{r-1}}{a\gamma_r(r-1)!}\int_{Q_1}^{P_1}\frac{x^{j-1}h^2(x)}{h'(x)}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx$$

which reduces to

(3.3)

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_{Q_1}^{P_1} x^{j-1}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\Big\{F(x)+Q_2-\frac{h^2(x)}{ah'(x)}f(x)\Big\}dx$$
$$=0.$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [10]) to equation (3.3), we get

$$F(x)=-Q_2+\frac{h^2(x)}{ah'(x)}f(x),\ \ P_1\le x\le Q_1$$

which proves that $f(x)$ has the form as in equation (1.7). □

Now we shall use recurrence relation in (2.1), $Q = 0$, to characterize the non-truncated general class of distribution by the following theorem.

THEOREM 3.2. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then*

(3.4)
$$E[X^{*j}(r,n,m,k)] = E[X^{*j}(r-1,n,m,k)] - \frac{j}{a\gamma_r}E[\phi(X^*(r,n,m,k))]$$

if and only if

$$F_1(x) = e^{-a/h(x)},\ x \in (\alpha,\ \beta).$$

Proof. The necessary part follows immediately from equation (2.1). On the other hand if the recurrence relation in equation (3.4) is satisfied, then on using equations (1.2), we have

(3.5)
$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_\alpha^\beta x^j[F_1(x)]^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx \\
&= \frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_\alpha^\beta x^j[F_1(x)]^{\gamma_r+m}f_1(x)g_m^{r-2}(F_1(x))dx \\
&\quad - \frac{jC_{r-1}}{a\gamma_r(r-1)!}\int_\alpha^\beta \frac{x^{j-1}h^2(x)}{h'(x)}[F_1(x)]^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx.
\end{aligned}$$

Integrating the first integral on the right hand side of equation (3.5), by parts, we get

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_\alpha^\beta x^j[F_1(x)]^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx \\
&= \frac{jC_{r-1}}{\gamma_r(r-1)!}\int_\alpha^\beta x^{j-1}[F_1(x)]^{\gamma_r}g_m^{r-1}(F_1(x))dx \\
&\quad + \frac{C_{r-1}}{(r-1)!}\int_\alpha^\beta x^j[F_1(x)]^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx \\
&\quad - \frac{jC_{r-1}}{a\gamma_r(r-1)!}\int_\alpha^\beta \frac{x^{j-1}h^2(x)}{h'(x)}[F_1(x)]^{\gamma_r-1}f_1(x)g_m^{r-1}(F_1(x))dx
\end{aligned}$$

which reduces to

(3.6)
$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_\alpha^\beta x^{j-1}[F_1(x)]^{\gamma_r-1}g_m^{r-1}(F_1(x))\Big\{F_1(x) - \frac{h^2(x)}{ah'(x)}f_1(x)\Big\}dx = 0.$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [10]) to equation (3.3), we get

$$\frac{f_1(x)}{F_1(x)} = \frac{ah'(x)}{h^2(x)}$$

which proves that

$$F_1(x) = e^{-a/h(x)},\ x \in (\alpha,\ \beta).$$

□

4. Conclusion

This paper deals with the lower generalized order statistics within a class of doubly truncated distributions. Some recurrence relations for single moments are derived. Two theorems for characterizing the general form of distribution based on single moments of lower generalized order statistics are given. Special cases are also deduced.

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