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SOME EXAMPLES OF THE UNION OF TWO LINEAR STAR-CONFIGURATIONS IN $\mathbb{P}^2$ HAVING GENERIC HILBERT FUNCTION

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Abstract. In [20] and [22], the author proved that the union of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$ with $3 \le t \le 10$ and $t \le s$ has generic Hilbert function. In this paper, we prove that the union of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$ with $3 \le t$ and $\binom{t}{2} - 1 \le s$ has also generic Hilbert function.

1. Introduction

Let $R = \Bbbk[x_0, x_1, \dots, x_n]$ be an $(n+1)$-variable polynomial ring and $A = R/I$ where I is a homogeneous ideal in R. Then $A = \bigoplus_{i=0}^{\infty} A_i$ is also a graded ring. In this situation the *Hilbert function of A* is the function

$$\mathbf{H}(A, i) := \dim_{\Bbbk} A_i = \dim_{\Bbbk} R_i - \dim_{\Bbbk} I_i = \binom{i+n}{n} - \dim_{\Bbbk} I_i.$$

If $I := I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote the Hilbert function of $\mathbb{X}$ by

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}(R/I_{\mathbb{X}}, t)$$

(see [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13]). In particluar, If $\mathbb{X}$ is a subscheme in $\mathbb{P}^2$ and

$$\mathbf{H}_{\mathbb{X}}(d) = \min\left\{\binom{d+2}{2},\ \deg(\mathbb{X})\right\}$$

for every $d \ge 0$, then we say that $\mathbb{X}$ has *generic Hilbert function.*

In this paper, we study the union of two star-configurations in $\mathbb{P}^2$ defined by general forms (see also [2, 20, 21, 22]). In [21], the author found conditions for a star-configuration in $\mathbb{P}^2$ to have generic Hilbert function based on the degrees of these general forms. In [2, 21], the

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authors also found conditions when a graded Artinian ring $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ has the Weak Lefschetz property for two star-configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^2$ (see also [14, 15, 16, 17, 18, 19]).

The following proposition in [3] is about the ideal of general forms in R, which leads to the definition of a *star-configuration* and a *linear star-configuration* in $\mathbb{P}^n$.

Proposition 1.1. [3, Proposition 2.1] *Let $F_1, F_2, \dots, F_s$ be general forms in $R = \Bbbk[x_0, x_1, \dots, x_n]$ with $s \geq 3$. Then*

$$\bigcap_{1\leq i<j\leq s}(F_i, F_j) = (\tilde{F}_1, \dots, \tilde{F}_s), \text{ where } \tilde{F}_i = \frac{\prod_{j=1}^{s} F_j}{F_i} \text{ for } i = 1, \dots, s.$$

The variety $\mathbb{X}$ in $\mathbb{P}^n$ of the ideal $\bigcap_{1\leq i<j\leq s}(F_i, F_j) = (\tilde{F}_1, \dots, \tilde{F}_s)$ in Proposition 1.1 is called a *star-configuration* in $\mathbb{P}^n$ of type s. Furthermore, if the F_i are all general linear forms in R, the star-configuration $\mathbb{X}$ is called a *linear star-configuration* in $\mathbb{P}^n$.

In this paper, if $\mathbb{X} := \mathbb{X}^{(t,s)}$ is the union of two linear star-configurations $\mathbb{X}_1$ and $\mathbb{X}_2$ in $\mathbb{P}^2$ of types t and s (type $t \times s$ for short), then $\mathbb{X}$ has generic Hilbert function for $3 \leq t$ and $\binom{t}{2} - 1 \leq s$. Moreover, we also show that $\sigma(\mathbb{X}) = s$ for such t and s, where $\sigma(\mathbb{X}) := \min\{d \mid \mathbf{H}_{\mathbb{X}}(d-1) = \mathbf{H}_{\mathbb{X}}(d)\}$.

In Section 3, we propose some questions for further study.

2. The union of two linear star-configurations in $\mathbb{P}^2$

Before we start to prove the main theorem, we introduce some notations for convenience. Let $L_1, \dots, L_{s-1}, L_s$, and $M_1, \dots, M_t$ be general linear forms for $s \geq 3$ and $t \geq 3$, respectively. Define

$\mathbb{X}_1 = \mathbb{Y}_1$ is a linear star-configuration in $\mathbb{P}^2$ defined by $M_1, \dots, M_t$,
$\mathbb{X}_2$ is a linear star-configuration in $\mathbb{P}^2$ defined by $L_1, \dots, L_{s-1}, L_s$,
$\mathbb{Y}_2 \subseteq \mathbb{X}_2$ is a linear star-configuration in $\mathbb{P}^2$ defined by $L_1, \dots, L_{s-1}$.
$\mathbb{Y} := \mathbb{X}^{(t,s-1)} := \mathbb{Y}_1 \cup \mathbb{Y}_2,\ \mathbb{X} := \mathbb{X}^{(t,s)} := \mathbb{X}_1 \cup \mathbb{X}_2,$ and
$G_{s-1} := L_1 \cdots L_{s-1},$ respectively.

The first idea is that if $\mathbb{X}'$ is the union of two finite sets of points defined by linear forms $M_1, \dots, M_t$ and $L_1, L_2, \dots, L_s$ in R (not necessarily general), respectively, then the points in $\mathbb{X}$ are more general than the points in $\mathbb{X}'$. This implies for every $i \geq 0$ we get

$$\mathbf{H}_{\mathbb{X}'}(i) \leq \mathbf{H}_{\mathbb{X}}(i).$$

The second idea is using *Bezout*'s Theorem in $\mathbb{P}^2$ to find the union $\mathbb{X}'$ of two sets of points defined by linear forms $M_1, \dots, M_t$ and $L_1, L_2, \dots, L_s$

in R, respectively, such that

$$\mathbf{H}_{\mathbb{X}}(i) = \mathbf{H}_{\mathbb{X}'}(i) = \min\left\{|\mathbb{X}|, \binom{i+2}{2}\right\} \quad \text{for some} \quad i \geq 0.$$

In other words, if a form F of degree d in R vanishes on $(d+1)$-points on a line defined by a linear form M in R, then F is divided by a linear form M. Throughout this section, we shall not distinguish $\mathbb{X}$ from $\mathbb{X}'$ for convenience.

Proposition 2.1. *With notation as above, $\mathbb{X} := \mathbb{X}^{(t,s)}$ has generic Hilbert function and $\sigma(\mathbb{X}) = s$ for $s \geq \binom{t}{2}$ and $t \geq 3$.*

Proof. We shall prove this by induction on s. First, let $s = \binom{t}{2}$, and assume that $\mathbb{X} := \mathbb{X}_1 \cup \mathbb{X}_2$ where $\mathbb{X}_1$ and $\mathbb{X}_2$ are linear star-configurations in $\mathbb{P}^2$ defined by general linear forms $M_1, \ldots, M_t$ and $L_1, \ldots, L_s$, respectively. Let $\mathbb{X}_1 := \{Q_1, \ldots, Q_s\}$. Without loss of generality, we may assume that L_i vanishes on a point Q_i for $i = 1, \ldots, s-1$. If $F \in (I_{\mathbb{X}})_{s-1}$ then, by Bezóut's Theorem,

$$F = \alpha L_1 \cdots L_{s-1}$$

for some $\alpha \in k$. Moreover, since F also vanishes on the point Q_s, which none of $L_1, \ldots, L_{s-1}$ vanishes, we get that $F = 0$, that is , $(I_{\mathbb{X}})_{s-1} = 0$. Hence

$$\mathbf{H}(R/I_{\mathbb{X}}, s-1) = \binom{s+1}{2} = \binom{s}{2} + s = \binom{s}{2} + \binom{t}{2} = \deg(\mathbb{X}),$$

and so $\mathbb{X}$ has generic Hilbert function as

$$\begin{array}{cccccccccc} \mathbf{H}_{\mathbb{X}} & : & 1 & \binom{3}{2} & \cdots & \binom{(s-3)+2}{2} & \binom{(s-2)+2}{2} & \binom{(s-1)+2}{2} & \binom{(s-1)+2}{2} & \rightarrow, \\ &&&&&&& \| && \\ &&&&&&& \deg(\mathbb{X}) && \end{array}$$

and $\sigma(\mathbb{X}) = s$, as we wished.

Now suppose $s > \binom{t}{2}$. Let $\mathbb{Y} := \mathbb{X}^{(t,s-1)}$ be the union of two linear star-configurations in $\mathbb{P}^2$ defined by linear forms $M_1, \ldots, M_t$ and $L_1, \ldots, L_{s-1}$, respectively. Now we consider the following equations:

$$\begin{array}{rcccccccc} &&&&& (s-2)\text{-nd} &&& \\ \mathbf{H}(R/I_{\mathbb{X}}, -) & : & 1 & \binom{1+2}{2} & \cdots & - & \binom{s}{2} + \binom{t}{2} & \rightarrow, \\ \mathbf{H}(R/I_{\mathbb{Y}}, -) & : & 1 & \binom{1+2}{2} & \cdots & \binom{s-1}{2} + \binom{t}{2} & \binom{s-1}{2} + \binom{t}{2} & \rightarrow, \\ \mathbf{H}(R/(L_s, G_{s-1}), -) & : & 1 & 2 & \cdots & s-1 & s-1 & \rightarrow, \\ \mathbf{H}(R/(I_{\mathbb{Y}}, L_s, G_{s-1}), -) & : & 1 & 2 & \cdots & - & 0 & \rightarrow, \\ \mathbf{H}(R/(I_{\mathbb{Y}}, L_s), -) & : & 1 & 2 & \cdots & \binom{t}{2} & 0 & \rightarrow. \end{array}$$

Since $\deg G_{s-1} = s-1$, we have

$$\mathbf{H}(R/(I_{\mathbb{Y}}, L_s, G_{s-1}), s-2) = \mathbf{H}(R/(I_{\mathbb{Y}}, L_s), s-2) = \binom{t}{2},$$

and thus

$$\begin{aligned}&\mathbf{H}(R/I_{\mathbb{X}}, s-2)\\ &= \mathbf{H}(R/I_{\mathbb{Y}}, s-2) + \mathbf{H}(R/(L_s, G_{s-1}), s-2) - \mathbf{H}(R/(I_{\mathbb{Y}}, L_s, G_{s-1}), s-2)\\ &= \binom{(s-3)+2}{2} + \binom{t}{2} + (s-1) - \binom{t}{2} = \binom{(s-3)+2}{2} + (s-1)\\ &= \binom{(s-2)+2}{2}.\end{aligned}$$

This means that $\mathbb{X}$ has generic Hilbert function as

$$\mathbf{H}_{\mathbb{X}} \;:\; 1 \quad \binom{1+2}{2} \quad \cdots \quad \binom{(s-3)+2}{2} \quad \binom{(s-2)+2}{2} \quad \binom{s}{2}+\binom{t}{2} \quad \binom{s}{2}+\binom{t}{2} \quad \rightarrow,$$

and $\sigma(\mathbb{X}) = s$, which completes the proof. □

Corollary 2.2. *With notation as above, $\mathbb{X} := \mathbb{X}^{(t,s-1)}$ has generic Hilbert function and $\sigma(\mathbb{X}) = s$ for $s = \binom{t}{2}$ and $t \geq 3$.*

Proof. Note that, by Proposition 2.1, $\mathbb{Z} := \mathbb{X}^{(t,s)}$ has generic Hilbert function, and so we get the following equation.

$$\begin{array}{rclllll} & & & & & (s-1)\text{-st} & & \\ \mathbf{H}(R/I_{\mathbb{Z}}, -) & : & 1 & \binom{1+2}{2} & \cdots & \binom{s}{2}+\binom{t}{2} & \binom{s}{2}+\binom{t}{2} & \rightarrow, \\ \mathbf{H}(R/I_{\mathbb{X}}, -) & : & 1 & \binom{1+2}{2} & \cdots & \binom{s-1}{2}+\binom{t}{2} & \binom{s-1}{2}+\binom{t}{2} & \rightarrow, \\ \mathbf{H}(R/(L_s, G_{s-1}), -) & : & 1 & 2 & \cdots & s-1 & s-1 & \rightarrow, \\ \mathbf{H}(R/(I_{\mathbb{X}}, L_s, G_{s-1}), -) & : & 1 & 2 & \cdots & 0 & 0 & \rightarrow, \\ \mathbf{H}(R/(I_{\mathbb{X}}, L_s), -) & : & 1 & 2 & \cdots & - & 0 & \rightarrow. \end{array}$$

Let $F \in (I_{\mathbb{X}})_{s-2}$ and let $\mathbb{X}_1 := \{Q_1, \dots, Q_s\}$. Without loss of generality, we assume that

$$\begin{array}{lll} L_1 & \text{vanishes on } (s-1)\text{-points} & P_{1,2}, \dots, P_{1,s-1}, Q_1, \\ L_2 & \text{vanishes on } (s-2)\text{-points} & P_{2,3}, \dots, P_{2,s-1}, Q_2, \\ & \vdots & \\ L_{t-1} & \text{vanishes on } (s-t+1)\text{-points} & P_{t-1,t}, \dots, P_{t-1,s-1}, Q_{t-1}, \\ & \vdots & \\ L_{s-3} & \text{vanishes on 3-points} & P_{s-3,s-2}, P_{s-3,s-1}, Q_{s-3}, \\ L_{s-2} & \text{vanishes on 2-points} & P_{s-2,s-1}, Q_{s-2}, \end{array}$$

where $P_{i,j}$ is the point defined by two linear forms L_i and L_j for $i < j$. Then, by Bezóut's theorem, $F = \alpha L_1 \cdots L_{s-2}$. Moreover, since F has to vanish on two more points Q_{s-1} and Q_s, we see that $F = 0$, that is, $(I_{\mathbb{X}})_{s-2} = 0$. It follows that $\mathbb{X}$ has generic Hilbert function

$$\mathbf{H}(R/I_{\mathbb{X}}, -) \;:\; 1 \quad 3 \quad \cdots \quad \binom{(s-2)+2}{2} \quad \binom{s-1}{2}+\binom{t}{2} \quad \binom{s-1}{2}+\binom{t}{2} \quad \rightarrow,$$

and $\sigma(\mathbb{X}) = s$, as we wished. □

3. Additional comments and questions

In [4], the authors proved that the secant variety $\mathrm{Sec}_{s-1}(\mathrm{Split}_d(\mathbb{P}^n))$ to the variety $\mathrm{Split}_d(\mathbb{P}^n)$ of split forms in $R = \Bbbk[x_0, x_1, \ldots, x_n]$ is not defective for $3(s-1) \le n$ and $2 < d$ (see also [5]). Moreover, in [20], the author proved that the secant variety $\mathrm{Sec}_1(\mathrm{Split}_d(\mathbb{P}^2))$ to the variety $\mathrm{Split}_d(\mathbb{P}^2)$ of split forms in $R = \Bbbk[x_0, x_1, x_2]$ is not defective for $2 < d$, which is not covered by the result of [4], calculating the Hilbert function of two linear star-configurations in $\mathbb{P}^2$ of type $d \times d$ with $d > 2$.

In particular, in [20, 22], the author found that the union of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$ has generic Hilbert function for $3 \le t \le 10$ and $t \le s$, and we also found that some different type of the union of two linear star-configurations in $\mathbb{P}^2$ has also generic Hilbert function (see Proposition 2.1 and Corollary 2.2). Hence it is natural to ask the following question.

Question 3.1. Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be star-configurations in $\mathbb{P}^2$ defined by s-general forms of degrees $1 \le d_1 \le \cdots \le d_s$ with $3 \le s$, respectively, and let $\mathbb{X} := \mathbb{X}_1 \cup \mathbb{X}_2$.

(a) Does $\mathbb{X}$ have generic Hilbert function in general?
(b) Does $\mathbb{X}$ have generic Hilbert function if $1 \le d_1 = \cdots = d_s$?
(c) Does $\mathbb{X}$ have generic Hilbert function if $1 = d_1 = \cdots = d_s$?

In fact, Question 3.1 (a) is not true in general. Here is an example.

Example 3.2. Let $L_i, M_j \in R_1$ for $i, j = 1, \ldots, 5$ and $F, G \in R_5$. Assume $\mathbb{X}$ is the union of two star-configurations in $\mathbb{P}^2$ defined by 6-forms $L_1, \ldots, L_5, F$ and $M_1, \ldots, M_5, G$, respectively. Then there exists one generator $L_1 \ldots L_5 M_1 \ldots M_5 \in (I_{\mathbb{X}})_{10}$, and hence, by Proposition 1.1, the Hilbert function of $\mathbb{X}$ is of the form

$$\mathbf{H}_{\mathbb{X}} \ : \ 1 \quad \binom{1+2}{2} \quad \cdots \quad \binom{9+2}{2} \quad \binom{10+2}{2} - 1 \quad \cdots,$$

which indicates $\mathbf{H}_{\mathbb{X}}(10) = 65 \neq 70 = \deg(\mathbb{X})$. Thus, $\mathbb{X}$ does not have generic Hilbert function.

Indeed, we can generalize Example 3.2 as follows:

Remark 3.3. Let $L_1, \ldots, L_{s-1}, M_1, \ldots, M_{s-1} \in R_1$ and $F, G \in R_c$ with $s \ge 6$ and $c \ge s-1$. Assume $\mathbb{X}$ is the union of two star-configurations $\mathbb{X}_1$ and $\mathbb{X}_2$ in $\mathbb{P}^2$ defined by s-forms $L_1, \ldots, L_{s-1}, F$ and $M_1, \ldots, M_{s-1}, G$, respectively. Since the ideal $I_{\mathbb{X}}$ has one generator $L_1 \cdots L_{s-1} M_1 \cdots M_{s-1}$ in degree $d = 2(s-1)$, the Hilbert function of $\mathbb{X}$

is of the form

$$\mathbf{H}_{\mathbb{X}} \ : \ 1 \quad \binom{1+2}{2} \quad \cdots \quad \binom{(2s-3)+2}{2} \quad \binom{2(s-1)+2}{2} - 1 \quad \cdots ,$$

and hence $\mathbf{H}_{\mathbb{X}}(d) < \binom{d+2}{2}$. Moreover, since $s \geq 6$, we also have that

$$\mathbf{H}_{\mathbb{X}}(d) < \binom{d+2}{2} < \deg(\mathbb{X}),$$

which follows that $\mathbb{X}$ does not have generic Hilbert function.

Note that if $\mathbb{X}$ is the union of two star-configurations in $\mathbb{P}^2$ defined by forms of degrees $1, 1, 1, 1, 4$, then $\mathbb{X}$ has generic Hilbert function as

$$\mathbf{H}_{\mathbb{X}} \ : \ 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 28 \quad 36 \quad 44 \quad \rightarrow .$$

However, we don't have any counter example to Question 3.1 (b) and (c) yet.

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