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POISSON BRACKETS DETERMINED BY JACOBIANS

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Abstract. Fix $n-2$ elements $h_1, \cdots, h_{n-2}$ of the quotient field B of the polynomial algebra $\mathbb{C}[x_1, x_2, \cdots, x_n]$. It is proved that B is a Poisson algebra with Poisson bracket defined by $\{f, g\} = \det(\mathrm{Jac}(f, g, h_1, \cdots, h_{n-2}))$ for any $f, g \in B$, where $\det(\mathrm{Jac})$ is the determinant of a Jacobian matrix.

In [1], Jordan and the author studied Poisson brackets on the polynomial algebra $\mathbb{C}[x, y, z]$ in three indeterminates x, y, z determined by Jacobians. In particular, for an arbitrary rational function $s/t \in \mathbb{C}(x, y, z)$, they analyzed the Poisson bracket determined by the formula

$$(\{x, y\}, \{y, z\}, \{z, x\}) = t^2 \nabla(s/t), \quad s/t \in \mathbb{C}(x, y, z), \tag{1}$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ is the gradient. The general rule is that, for $f, g \in \mathbb{C}[x, y, z]$,

$$\{f, g\} = t^2 \det(\mathrm{Jac}(f, g, s/t)),$$

where $\det(\mathrm{Jac}(f, g, s/t))$ denotes the determinant of the Jacobian matrix of $(f, g, s/t)$.

The purpose of this paper is to generalize the bracket (1) to the general polynomial algebra $A := \mathbb{C}[x_1, x_2, \cdots, x_n], n \geq 3$. For fixed $n-2$ elements $h_1, \cdots, h_{n-2}$ of the quotient field B of A, the fact that, for any $f, g \in B$, the bracket defined by

$$\{f, g\} = \det(\mathrm{Jac}(f, g, h_1, \cdots, h_{n-2})) \tag{2}$$

is a Poisson bracket is proved in [4] and [2, Theorem 1.4]. But the proof of [4] is not clear and that of [2] uses the Plücker relation and special derivations induced by the n-Jacobi identity in [3]. Here we prove by using only elementary algebraic theories that (2) is a Poisson bracket on B. Fix $s_1, t_1, \cdots, s_{n-2}, t_{n-2} \in A$ such that s_i and $t_i \neq 0$ are coprime

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for each $i = 1, 2, \cdots, n-2$. Next, as a corollary, we obtain that, for all $f, g \in A$,

$$\{f, g\} = (t_1 \cdots t_{n-2})^2 \det(\mathrm{Jac}(f, g, s_1/t_1, s_2/t_2, \cdots, s_{n-2}/t_{n-2})) \tag{3}$$

is a Poisson bracket in A, which is a generalization of (1). The presence of the factor $(t_1 \cdots t_{n-2})^2$ ensures that this restricts to a Poisson bracket on A.

Throughout the paper, A and B denote the algebra $\mathbb{C}[x_1, x_2, \cdots, x_n]$ with $n \geq 3$ and the quotient field of A respectively, as above.

NOTATION 1. Let $\mathcal{F} = (\varphi^{ij})$ be an $(n-2) \times n$-matrix with entries $\varphi^{ij} \in B$.

(a) For any $i, j = 1, \cdots, n$, denote by $\mathcal{F}_{ij}$ the determinant of the $n \times n$-matrix $\begin{pmatrix} e_i \\ e_j \\ \mathcal{F} \end{pmatrix}$, where $\{e_i\}_{i=1}^n$ is the standard basis of B^n.

(b) For $z \in B$, ∇z denotes the row vector of B^n

$$\nabla z = \left(\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \cdots, \frac{\partial z}{\partial x_n}\right) = \frac{\partial z}{\partial x_1} e_1 + \frac{\partial z}{\partial x_2} e_2 + \cdots + \frac{\partial z}{\partial x_n} e_n.$$

LEMMA 2. *For all $i, j = 1, \cdots, n$, $\mathcal{F}_{ij} = -\mathcal{F}_{ji}$. In particular, $\mathcal{F}_{ii} = 0$.*

Proof. It is clear from the elementary linear algebra. □

For any $(n-2) \times n$-matrix $\mathcal{F}$ in Notation 1 and for any $f, g \in B$, set

$$\{f, g\}^{\mathcal{F}} = \begin{vmatrix} \nabla f \\ \nabla g \\ \mathcal{F} \end{vmatrix} \in B. \tag{4}$$

Then the bilinear product $\{\cdot, \cdot\}^{\mathcal{F}}$ in (4) is antisymmetric and satisfies the Leibniz rule. Thus the algebra B with the bilinear product $\{\cdot, \cdot\}^{\mathcal{F}}$ becomes a Poisson algebra if and only if $\{\cdot, \cdot\}^{\mathcal{F}}$ satisfies the Jacobi identity. In such case, $\{\cdot, \cdot\}^{\mathcal{F}}$ is said to be a Poisson bracket in B.

NOTATION 3. For $a, b, c \in B$, let

$$J_{\mathcal{F}}(a, b, c) = \{\{a, b\}^{\mathcal{F}}, c\}^{\mathcal{F}} + \{\{b, c\}^{\mathcal{F}}, a\}^{\mathcal{F}} + \{\{c, a\}^{\mathcal{F}}, b\}^{\mathcal{F}}.$$

Thus a, b and c satisfy the Jacobi identity for $\{\cdot, \cdot\}^{\mathcal{F}}$ if and only if $J_{\mathcal{F}}(a, b, c) = 0$.

Note that any derivation on an algebra is determined by values of generators.

Proposition 4. [1, Proposition 1.14] *The algebra B is a Poisson algebra under $\{\cdot,\cdot\}^{\mathcal{F}}$ if and only if $J_{\mathcal{F}}(x_i,x_j,x_k)=0$ for any $1\le i,j,k\le n$.*

Lemma 5. *Let $1\le i,j,k\le n$. Then $J_{\mathcal{F}}(x_i,x_j,x_k)=0$ if and only if*

$$\sum_{\ell=1}^{n}\left[\frac{\partial\mathcal{F}_{ij}}{\partial x_\ell}\mathcal{F}_{\ell k}+\frac{\partial\mathcal{F}_{jk}}{\partial x_\ell}\mathcal{F}_{\ell i}+\frac{\partial\mathcal{F}_{ki}}{\partial x_\ell}\mathcal{F}_{\ell j}\right]=0. \tag{5}$$

Proof. If $i=j$, $i=k$ or $j=k$ then we have $J_{\mathcal{F}}(x_i,x_j,x_k)=0$ since $\{\cdot,\cdot\}^{\mathcal{F}}$ is antisymmetric, and (5) holds by Lemma 2. Hence we may assume that $i<j<k$. Observe that

$$\begin{aligned}
J_{\mathcal{F}}(x_i,x_j,x_k)&=\begin{vmatrix}\nabla\begin{vmatrix}e_i\\ e_j\\ \mathcal{F}\end{vmatrix}\\ e_k\\ \mathcal{F}\end{vmatrix}+\begin{vmatrix}\nabla\begin{vmatrix}e_j\\ e_k\\ \mathcal{F}\end{vmatrix}\\ e_i\\ \mathcal{F}\end{vmatrix}+\begin{vmatrix}\nabla\begin{vmatrix}e_k\\ e_i\\ \mathcal{F}\end{vmatrix}\\ e_j\\ \mathcal{F}\end{vmatrix}\\
&=\begin{vmatrix}\sum_{\ell=1}^{n}\frac{\partial\mathcal{F}_{ij}}{\partial x_\ell}e_\ell\\ e_k\\ \mathcal{F}\end{vmatrix}+\begin{vmatrix}\sum_{\ell=1}^{n}\frac{\partial\mathcal{F}_{jk}}{\partial x_\ell}e_\ell\\ e_i\\ \mathcal{F}\end{vmatrix}+\begin{vmatrix}\sum_{\ell=1}^{n}\frac{\partial\mathcal{F}_{ki}}{\partial x_\ell}e_\ell\\ e_j\\ \mathcal{F}\end{vmatrix}\\
&=\sum_{\ell=1}^{n}\left[\frac{\partial\mathcal{F}_{ij}}{\partial x_\ell}\mathcal{F}_{\ell k}+\frac{\partial\mathcal{F}_{jk}}{\partial x_\ell}\mathcal{F}_{\ell i}+\frac{\partial\mathcal{F}_{ki}}{\partial x_\ell}\mathcal{F}_{\ell j}\right].
\end{aligned}$$

Hence the result holds. □

Lemma 6. *For any $1\le i,j,k,\ell\le n$,*

$$\mathcal{F}_{ij}\mathcal{F}_{\ell k}+\mathcal{F}_{jk}\mathcal{F}_{\ell i}+\mathcal{F}_{ki}\mathcal{F}_{\ell j}=0. \tag{6}$$

Proof. If any two indices among i,j,k,ℓ are equal, say $\ell=i$, then

$$\mathcal{F}_{ij}\mathcal{F}_{\ell k}+\mathcal{F}_{jk}\mathcal{F}_{\ell i}+\mathcal{F}_{ki}\mathcal{F}_{\ell j}=\mathcal{F}_{ij}\mathcal{F}_{ik}+\mathcal{F}_{jk}\mathcal{F}_{ii}+\mathcal{F}_{ki}\mathcal{F}_{ij}=0$$

since $\mathcal{F}_{ii}=0$ and $\mathcal{F}_{ki}=-\mathcal{F}_{ik}$, and thus (6) holds in this case. So we may assume that $n\ge 4$ and that all i,j,k,ℓ are distinct. For any p,q such that $1\le p,q\le n$ and $p\ne q$, denote by X_{pq} the set of all bijective maps from $\{1,\cdots,n-2\}$ onto $\{1,\cdots,n\}\setminus\{p,q\}$ and, for $p=1,\cdots,n-2,q=1,\cdots,n$, denote by z_{pq} the (p,q)-entry of $\mathcal{F}$. The left hand of (6) is as follows:

$$\sum_{\sigma\in X_{ij},\tau\in X_{\ell k}}\left(\prod_{p=1}^{n-2} z_{p\sigma(p)}z_{p\tau(p)}\right)\begin{vmatrix} e_i\\ e_j\\ e_{\sigma(1)}\\ \vdots\\ e_{\sigma(n-2)}\end{vmatrix}\begin{vmatrix} e_\ell\\ e_k\\ e_{\tau(1)}\\ \vdots\\ e_{\tau(n-2)}\end{vmatrix}$$
$$+\sum_{\sigma\in X_{jk},\tau\in X_{\ell i}}\left(\prod_{p=1}^{n-2} z_{p\sigma(p)}z_{p\tau(p)}\right)\begin{vmatrix} e_j\\ e_k\\ e_{\sigma(1)}\\ \vdots\\ e_{\sigma(n-2)}\end{vmatrix}\begin{vmatrix} e_\ell\\ e_i\\ e_{\tau(1)}\\ \vdots\\ e_{\tau(n-2)}\end{vmatrix}$$
$$+\sum_{\sigma\in X_{ki},\tau\in X_{\ell j}}\left(\prod_{p=1}^{n-2} z_{p\sigma(p)}z_{p\tau(p)}\right)\begin{vmatrix} e_k\\ e_i\\ e_{\sigma(1)}\\ \vdots\\ e_{\sigma(n-2)}\end{vmatrix}\begin{vmatrix} e_\ell\\ e_j\\ e_{\tau(1)}\\ \vdots\\ e_{\tau(n-2)}\end{vmatrix}.$$

Let $(\sigma,\tau)\in X_{ij}\times X_{\ell k}$. We will find a unique $(\mu,\nu)\in X_{jk}\times X_{\ell i}$ (or $(\mu,\nu)\in X_{ki}\times X_{\ell j}$) such that $z_{p\sigma(p)}z_{p\tau(p)} = z_{p\mu(p)}z_{p\nu(p)}$ for each $p = 1,\cdots,n-2$ and that $\begin{vmatrix} e_j\\ e_k\\ e_{\mu(1)}\\ \vdots\\ e_{\mu(n-2)}\end{vmatrix}\begin{vmatrix} e_\ell\\ e_i\\ e_{\nu(1)}\\ \vdots\\ e_{\nu(n-2)}\end{vmatrix}$ is nonzero. (If $(\mu,\nu)\in X_{ki}\times X_{\ell j}$ then $\begin{vmatrix} e_k\\ e_i\\ e_{\mu(1)}\\ \vdots\\ e_{\mu(n-2)}\end{vmatrix}\begin{vmatrix} e_\ell\\ e_j\\ e_{\nu(1)}\\ \vdots\\ e_{\nu(n-2)}\end{vmatrix}$ is nonzero.) There exists a unique p_1 such that $k=\sigma(p_1)$, and then choose $\tau(p_1)$. If $\tau(p_1)\neq i$ and $\tau(p_1)\neq j$ then there exists a unique p_2 such that $\tau(p_1)=\sigma(p_2)$. If $\tau(p_2)\neq i$ and $\tau(p_2)\neq j$ then choose p_3 such that $\tau(p_2)=\sigma(p_3)$. This process stops only when $\tau(p_r)=i$ or $\tau(p_r)=j$, say $\tau(p_r)=i$, since there does not exist p such that $\sigma(p)=i$ or j. Hence we get a unique sequence

$$k=\sigma(p_1),\tau(p_1)=\sigma(p_2),\tau(p_2)=\sigma(p_3),\cdots,\tau(p_r)=i.$$

Note that all terms of the sequence are different since all $\sigma(p_1), \cdots, \sigma(p_r), i$ are distinct. Set

$$\mu(q) = \begin{cases} \sigma(q) & q \neq p_m \text{ for all } m = 1, \cdots, r \\ \tau(p_m) & q = p_m \end{cases},$$

$$\nu(q) = \begin{cases} \tau(q) & q \neq p_m \text{ for all } m = 1, \cdots, r \\ \sigma(p_m) & q = p_m. \end{cases}$$

Then $\mu \in X_{jk}$ and $\nu \in X_{\ell i}$ and it is easy to see that $z_{p\sigma(p)}z_{p\tau(p)} = z_{p\mu(p)}z_{p\nu(p)}$ for each $p = 1, \cdots, n-2$ and each row of the matrices $\begin{pmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{pmatrix}$ and $\begin{pmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{pmatrix}$ is different, as claimed. Changing suitable rows in the matrices

$$\begin{pmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{pmatrix} = \begin{pmatrix} e_j \\ e_k \\ e_{\sigma(1)} \\ \vdots \\ e_{\tau(p_1)} = e_{\sigma(p_2)} \\ \vdots \\ e_{\tau(p_m)} = e_{\sigma(p_{m+1})} \\ \vdots \\ e_{\sigma(n-2)} \end{pmatrix} \text{ and } \begin{pmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{pmatrix} = \begin{pmatrix} e_\ell \\ e_i \\ e_{\tau(1)} \\ \vdots \\ e_{\sigma(p_1)} = e_k \\ \vdots \\ e_{\sigma(p_m)} = e_{\tau(p_{m-1})} \\ \vdots \\ e_{\tau(n-2)} \end{pmatrix},$$

we have $\begin{vmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{vmatrix} = (-1)^{r+1} \begin{vmatrix} e_i \\ e_j \\ e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n-2)} \end{vmatrix}$ and $\begin{vmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{vmatrix} = (-1)^{r} \begin{vmatrix} e_\ell \\ e_k \\ e_{\tau(1)} \\ \vdots \\ e_{\tau(n-2)} \end{vmatrix}$,

hence

$$\begin{vmatrix} e_i \\ e_j \\ e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_k \\ e_{\tau(1)} \\ \vdots \\ e_{\tau(n-2)} \end{vmatrix} = - \begin{vmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{vmatrix}.$$

Therefore (6) holds. □

Lemma 7. *For any* $1 \leq i, j, k \leq n$, $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ *if and only if*

$$\sum_{\ell=1}^{n} \left[\mathcal{F}_{ij} \frac{\partial \mathcal{F}_{\ell k}}{\partial x_\ell} + \mathcal{F}_{jk} \frac{\partial \mathcal{F}_{\ell i}}{\partial x_\ell} + \mathcal{F}_{ki} \frac{\partial \mathcal{F}_{\ell j}}{\partial x_\ell} \right] = 0. \tag{7}$$

Proof. Since

$$\begin{aligned}\frac{\partial \mathcal{F}_{ij}}{\partial x_\ell} \mathcal{F}_{\ell k} + \frac{\partial \mathcal{F}_{jk}}{\partial x_\ell} \mathcal{F}_{\ell i} + \frac{\partial \mathcal{F}_{ki}}{\partial x_\ell} \mathcal{F}_{\ell j} &= \frac{\partial}{\partial x_\ell} \left(\mathcal{F}_{ij} F_{\ell k} + \mathcal{F}_{jk} \mathcal{F}_{\ell i} + \mathcal{F}_{ki} \mathcal{F}_{\ell j} \right) \\ &\quad - \left[\mathcal{F}_{ij} \frac{\partial \mathcal{F}_{\ell k}}{\partial x_\ell} + \mathcal{F}_{jk} \frac{\partial \mathcal{F}_{\ell i}}{\partial x_\ell} + \mathcal{F}_{ki} \frac{\partial \mathcal{F}_{\ell j}}{\partial x_\ell} \right],\end{aligned}$$

(7) follows from (5) and (6). □

Lemma 8. *For any* $\varphi^1, \cdots, \varphi^{n-2} \in B$, *let* $\mathcal{F} = \begin{pmatrix} \nabla \varphi^1 \\ \vdots \\ \nabla \varphi^{n-2} \end{pmatrix}$. *Then* $\sum_{\ell=1}^{n} \frac{\partial \mathcal{F}_{i\ell}}{\partial x_\ell} = 0$ *for each* $i = 1, \cdots, n$.

Proof. We may assume that $i \neq \ell$ since $\mathcal{F}_{ii} = 0$. Denote by $X_{i\ell}$ the set of all bijective maps from $\{1, \cdots, n-2\}$ onto $\{1, \cdots, n\} \setminus \{i, \ell\}$ as in the proof of Lemma 6 and set

$$< i, \ell >= \begin{cases} i + \ell - 1 & \text{if } i < \ell, \\ i + \ell & \text{if } i > \ell. \end{cases}$$

Then we have

$$\begin{aligned} &\sum_{\ell=1, \ell \neq i}^{n} \frac{\partial \mathcal{F}_{i\ell}}{\partial x_\ell} \\ &= \sum_{\ell=1, \ell \neq i}^{n} \sum_{\sigma \in X_{i\ell}} (-1)^{<i,\ell>} \mathrm{sgn}(\sigma) \frac{\partial}{\partial x_\ell} \left(\frac{\partial \varphi^1}{\partial x_{\sigma(1)}} \frac{\partial \varphi^2}{\partial x_{\sigma(2)}} \cdots \frac{\partial \varphi^{n-2}}{\partial x_{\sigma(n-2)}} \right) \\ &= \sum_{\ell=1, \ell \neq i}^{n} \sum_{\sigma \in X_{i\ell}} (-1)^{<i,\ell>} \mathrm{sgn}(\sigma) \left[\frac{\partial^2 \varphi^1}{\partial x_\ell \partial x_{\sigma(1)}} \left(\frac{\partial \varphi^2}{\partial x_{\sigma(2)}} \cdots \frac{\partial \varphi^{n-2}}{\partial x_{\sigma(n-2)}} \right) + \cdots \right. \\ &\qquad \left. + \left(\frac{\partial \varphi^1}{\partial x_{\sigma(1)}} \cdots \frac{\partial \varphi^{n-3}}{\partial x_{\sigma(n-3)}} \right) \frac{\partial^2 \varphi^{n-2}}{\partial x_\ell \partial x_{\sigma(n-2)}} \right]. \end{aligned}$$

Fix a term

$$A = \left(\frac{\partial \varphi^1}{\partial x_{\sigma(1)}} \cdots \frac{\partial \varphi^{k-1}}{\partial x_{\sigma(k-1)}} \right) \frac{\partial^2 \varphi^k}{\partial x_\ell \partial x_{\sigma(k)}} \left(\frac{\partial \varphi^{k+1}}{\partial x_{\sigma(k+1)}} \cdots \frac{\partial \varphi^{n-2}}{\partial x_{\sigma(n-2)}} \right).$$

Let $\sigma(k) = j$. Define $\tau \in X_{ij}$ by $\tau(k) = \ell, \tau(q) = \sigma(q)$ for all $q \neq k$. Then the term

$$B = \left(\frac{\partial\varphi^1}{\partial x_{\tau(1)}} \cdots \frac{\partial\varphi^{k-1}}{\partial x_{\tau(k-1)}}\right) \frac{\partial^2\varphi^k}{\partial x_j \partial x_{\tau(k)}} \left(\frac{\partial\varphi^{k+1}}{\partial x_{\tau(k+1)}} \cdots \frac{\partial\varphi^{n-2}}{\partial x_{\tau(n-2)}}\right)$$

is equal to A since $\frac{\partial^2\varphi^k}{\partial x_\ell \partial x_j} = \frac{\partial^2\varphi^k}{\partial x_j \partial x_\ell}$ for all ℓ, j, k, and the coefficients of A and B are $(-1)^{<i,\ell>}\mathrm{sgn}(\sigma)$ and $(-1)^{<i,\ell>}\mathrm{sgn}(\tau)$, respectively. Let $|\ell - j| = m$. We may assume that $j < \ell < i$. Thus $\ell = j + m$. Let $\sigma(k_1) = \sigma(k) = j, \sigma(k_2) = j+1, \cdots, \sigma(k_m) = j + m - 1$. Hence $\tau(k_1) = \ell, \tau(k_2) = j+1, \cdots, \tau(k_m) = j + m - 1$. Arrange elements of $X_{i\ell}$ and X_{ij} by using order relation:

$$\begin{aligned} X_{i\ell} &= \{p_1 < p_2 < \cdots < p_{n-2}\} \\ &= \{\cdots, j-1, \quad j, \cdots, \qquad j+m-1 = \ell - 1, \quad \ell+1, \cdots\} \\ X_{ij} &= \{q_1 < q_2 < \cdots < q_{n-2}\} \\ &= \{\cdots, j-1, j+1, \cdots, \qquad\qquad \ell, \qquad\qquad \ell+1, \cdots\}. \end{aligned}$$

Identifying p_s and q_s to s for all $s = 1, \cdots, n-2$, σ and τ are permutations in $\{1, \cdots, n-2\}$ and $\tau^{-1}\sigma$ is defined as follows:

$$\tau^{-1}\sigma(k_1) = k_2, \tau^{-1}\sigma(k_2) = k_3, \tau^{-1}\sigma(k_3) = k_4, \cdots, \tau^{-1}\sigma(k_m) = k_1$$
$$\tau^{-1}\sigma(p) = p \text{ for all } p \neq k_q, q = 1, \cdots, m.$$

Thus $\tau^{-1}\sigma$ is the cycle $(k_1 k_2 \cdots k_m)$ in the set $\{1, \cdots, n-2\}$. Hence $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\tau)$ if and only if $\ell - j = m$ is odd since $\mathrm{sgn}(\tau) = \mathrm{sgn}(\tau^{-1})$. It follows that the coefficient $(-1)^{<i,\ell>}\mathrm{sgn}(\sigma)$ of A is $-(-1)^{<i,j>}\mathrm{sgn}(\tau)$ and thus $\sum_{\ell=1,\ell\neq i}^n \frac{\partial \mathcal{F}_{i\ell}}{\partial x_\ell} = 0$. □

Theorem 9. *For any $\varphi^1, \cdots, \varphi^{n-2} \in B$, let* $\mathcal{F} = \begin{pmatrix} \nabla\varphi^1 \\ \vdots \\ \nabla\varphi^{n-2} \end{pmatrix}$. *Then B is a Poisson algebra under $\{\cdot,\cdot\}^{\mathcal{F}}$.*

Proof. It is enough to show that $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ for all $1 \leq i, j, k \leq n$ by Proposition 4. By Lemma 8, (7) holds since $\mathcal{F}_{i\ell} = -\mathcal{F}_{\ell i}$, and thus we have $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ for all $1 \leq i, j, k \leq n$ by Lemma 7. □

Lemma 10. *Let $\mathcal{F}$ and $\mathcal{G}$ be $(n-2) \times n$-matrices in B such that $\mathcal{G}_i = a_i \mathcal{F}_i$ for each i, where $a_i \in B$ and $\mathcal{F}_i$ and $\mathcal{G}_i$ are the i-rows of $\mathcal{F}$ and $\mathcal{G}$ respectively. If $\{\cdot,\cdot\}^{\mathcal{F}}$ is a Poisson bracket on B then $\{\cdot,\cdot\}^{\mathcal{G}}$ is also a Poisson bracket and $\{\cdot,\cdot\}^{\mathcal{G}} = (a_1 \cdots a_{n-2})\{\cdot,\cdot\}^{\mathcal{F}}$.*

Proof. Set $a = a_1 \cdots a_{n-2}$. Since $\mathcal{G}_{ij} = a\mathcal{F}_{ij}$ for all i, j, we have

$$\frac{\partial \mathcal{G}_{ij}}{\partial x_\ell}\mathcal{G}_{\ell k} + \frac{\partial \mathcal{G}_{jk}}{\partial x_\ell}\mathcal{G}_{\ell i} + \frac{\partial \mathcal{G}_{ki}}{\partial x_\ell}\mathcal{G}_{\ell j} = a^2\left[\frac{\partial \mathcal{F}_{ij}}{\partial x_\ell}\mathcal{F}_{\ell k} + \frac{\partial \mathcal{F}_{jk}}{\partial x_\ell}\mathcal{F}_{\ell i} + \frac{\partial \mathcal{F}_{ki}}{\partial x_\ell}\mathcal{F}_{\ell j}\right] + a\frac{\partial a}{\partial x_\ell}\left(\mathcal{F}_{ij}F_{\ell k} + \mathcal{F}_{jk}\mathcal{F}_{\ell i} + \mathcal{F}_{ki}\mathcal{F}_{\ell j}\right).$$

Thus $\{\cdot,\cdot\}^{\mathcal{G}}$ is a Poisson bracket by (5) and (6). □

Corollary 11. *Fix $s_1, t_1, \cdots, s_{n-2}, t_{n-2} \in A$ such that s_i and $t_i \neq 0$ are coprime for each $i = 1, 2, \cdots, n-2$. Then (3) is a Poisson bracket on A.*

Proof. Under the notation of Theorem 9 and Lemma 10, set $\varphi^i = s_i/t_i$ and $a_i = t_i^2$ for all $i = 1, \cdots, n-2$. Then the result follows immediately by Theorem 9 and Lemma 10 since each component of $t_i^2\nabla(s_i/t_i)$ is an element of A. □

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