

CATEGORY OF MAPS AND GOTTLIEB SETS FOR MAPS, AND THEIR DUALS

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ABSTRACT. In this paper, we introduce and study the concepts of WC_k^f -spaces with respect to spaces which are generalized concepts of C_k^f -spaces for maps, and introduce the dual concepts of WC_k^f -spaces with respect to spaces and obtain some dual results.

1. Introduction

Throughout this paper, a space means a space of the homotopy type of a locally finite connected CW complex. All maps shall mean continuous functions. It is known that any space X is filtered by the projective spaces of ΩX by a result of Milnor [8] and Stasheff [10];

$$\Sigma\Omega X = P^1(\Omega X) \hookrightarrow P^2(\Omega X) \hookrightarrow \cdots \hookrightarrow P^\infty(\Omega X) \simeq X.$$

For each k , let $e_k^X : P^k(\Omega X) \rightarrow P^\infty(\Omega X) \simeq X$ be the natural inclusion. Let $f : A \rightarrow X$ be a map. A space X is called [5] a C_k^f -space if the inclusion $e_k^X : P^k(\Omega X) \rightarrow X$ is f -cyclic. It is known [5] that a space X is a C_k^f -space for a map $f : A \rightarrow X$ if and only if $G^f(Z, X) = [Z, X]$ for any space Z with $\text{cat } Z \leq k$. For any spaces Z, X , we define $\text{mapcat}(Z, X) \leq k$ if for any map $g : Z \rightarrow X$, $\text{cat } g \leq k$. It is known that if $\text{cat } Z \leq k$, then $\text{mapcat}(Z, X) \leq k$, but the converse does not hold (see Example 2.6).

In this paper, we introduce the concepts of WC_k^f -spaces with respect to spaces which are generalizations of C_k^f -spaces for maps [5] and study some properties of WC_k^f -spaces with respect to spaces. We show that for a space Z with $\text{mapcat}(Z, X) \leq k$, a space X is a WC_k^f -space with

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respect to Z if and only if $G^f(Z, X) = [Z, X]$. Let $f : A \rightarrow X$ and $g : B \rightarrow Y$ be any maps and Z a space with $\text{mapcat}(Z, X) \leq k$. Then we show that the product space $X \times Y$ is a $WC_k^{f \times g}$ -space with respect to Z if and only if X is a WC_k^f -space with respect to Z and Y is a WC_k^g -space with respect to Z . We also introduce the dual concepts of WC_k^f -spaces with respect to spaces and obtain some dual results.

2. WC_k^f -spaces with respect to spaces

The *LS category* of X [3], denoted $\text{cat } X$, is the least integer k such that X is the union of $k + 1$ open sets U_i , each contractible in X . We now recall the following Ganea's theorems [3].

THEOREM 2.1. ([3],[4]) *The category $\text{cat } X \leq k$ if and only if $e_k^X : P^k(\Omega X) \rightarrow X$ has a right homotopy inverse.*

The definition of LS category extends from spaces to continuous maps as follows. Let $g : X \rightarrow Y$ be a map. The *LS category of g* [3], denoted $\text{cat } g$ is the least integer k such that X is the union of $k + 1$ open sets U_i for which the restriction of g to each U_i is homotopic to a constant map $U_i \rightarrow *$. Note that $\text{cat } X = \text{cat } 1_X$.

THEOREM 2.2. [3] *Let $g : Z \rightarrow X$ be a map. Then the category $\text{cat } g \leq k$ if and only if there is a map $\bar{g} : Z \rightarrow P^k(\Omega X)$ such that $e_k^X \circ \bar{g} \sim g : Z \rightarrow X$, where $e_k^X : P^k(\Omega X) \rightarrow X$ is the natural inclusion.*

DEFINITION 2.3. *Let Z, X be any two spaces. The mapcategory of mapping space from Z to X is less than equal to k , $\text{mapcat}(Z, X) \leq k$, means that for any map $g : Z \rightarrow X$, $\text{cat } g \leq k$.*

It is clear that $\text{mapcat}(Z, X) \leq k$ if and only if $(e_k^X)_\# : [Z, P^k(\Omega X)] \rightarrow [Z, X]$ is an epimorphism.

The following propositions say that a relationship between category of a space and mapcategory of a mapping space.

PROPOSITION 2.4. *$\text{cat } Z \leq k$ if and only if for any space X , $\text{mapcat}(Z, X) \leq k$.*

Proof. Suppose that $\text{cat } Z \leq k$. Then there is a map $s_k^Z : Z \rightarrow P^k(\Omega Z)$ such that $e_k^Z \circ s_k^Z \sim 1$. Let X be a space and $g : Z \rightarrow X$ a map. We see $e_k^X \circ P^k(\Omega g) \sim g \circ e_k^Z$ by the naturality of the construction of

$P^k(\Omega Z)$ as is shown in the following homotopy commutative diagram;

$$\begin{array}{ccc} P^k(\Omega Z) & \xrightarrow{P^k(\Omega g)} & P^k(\Omega X) \\ e_k^Z \downarrow & & e_k^X \downarrow \\ Z & \xrightarrow{g} & X. \end{array}$$

Thus we have a map $\bar{g} = P^k(\Omega g) \circ s_k^Z : Z \rightarrow P^k(\Omega X)$ such that $e_k^X \circ \bar{g} \sim g$. Thus we know $\text{mapcat}(Z, Y) \leq k$. On the other hand, suppose that for any space X , the mapcategory $\text{mapcat}(Z, X) \leq k$. Taking $X = Z$ and $g = 1_Z$, then we know that $\text{cat } Z \leq k$. $\square$

PROPOSITION 2.5. *$\text{cat } X \leq k$ if and only if for any space Z , $\text{mapcat}(Z, X) \leq k$.*

Proof. Suppose that $\text{cat } X \leq k$. Then there is a map $s_k^X : X \rightarrow P^k(\Omega X)$ such that $e_k^X \circ s_k^X \sim 1$. Let Z be a space and $g : Z \rightarrow X$ a map. Then we have $e_k^X \circ (s_k^X \circ g) \sim 1_X \circ g \sim g$ and $\text{mapcat}(Z, X) \leq k$. On the other hand, suppose that for any space Z , the mapcategory $\text{mapcat}(Z, X) \leq k$. Taking $Z = X$ and $g = 1_X$, then we know that $\text{cat } X \leq k$. $\square$

In general, if $\text{cat } Z \leq k$, then $\text{mapcat}(Z, X) \leq k$ for a space X , but the converse does not hold by the following example.

EXAMPLE 2.6. It is well known fact that $\text{cat } \mathbb{C}P^n = n$. Thus if we take $X = \mathbb{C}P^k$ and $Z = \mathbb{C}P^{k+1}$, then we know, from Proposition 2.5, that $\text{mapcat}(Z, X) \leq k$, but $\text{cat } Z = k + 1$.

Let $f : A \rightarrow X$ be a map. A based map $g : B \rightarrow X$ is called *f-cyclic* [12] if there is a map $\phi : A \times B \rightarrow X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ j \uparrow & & \nabla \uparrow \\ A \vee B & \xrightarrow{(f \vee g)} & X \vee X \end{array}$$

is homotopy commute, where $j : A \vee B \rightarrow A \times B$ is the inclusion and $\nabla : X \vee X \rightarrow X$ is the folding map. Clearly, g is *f-cyclic* iff f is *g-cyclic*. In the case, $f = 1_X : X \rightarrow X$, $g : B \rightarrow X$ is called *cyclic* [15]. We denote the set of all homotopy classes of *f-cyclic* maps from B to X by $G^f(B, X) \subset [B, X]$ which is called the *Gottlieb set for a map $f : A \rightarrow X$* . If $f = 1_X : X \rightarrow X$, then we recover the *Gottlieb set* $G(B, X) = G^{1_X}(B, X)$ defined by Varadarajan [11]. In general,

$G(B, X) \subset G^f(B, X) \subset [B, X]$ for any spaces A, B, X and any map $f : A \rightarrow X$.

It is shown [14] that $G(S^5, S^5 \times S^5) \cong 2\mathbb{Z} \oplus 2\mathbb{Z} \neq G^{i_1}(S^5, S^5 \times S^5) \cong 2\mathbb{Z} \oplus \mathbb{Z} \neq [S^5, S^5 \times S^5] \cong \mathbb{Z} \oplus \mathbb{Z}$. It is introduced [5] that a space X is called a C_k^f -space if the inclusion $e_k^X : P^k(\Omega X) \rightarrow X$ is f -cyclic.

LEMMA 2.7. *Let $f : A \rightarrow X$ be a map. Then $g : B \rightarrow X$ is f -cyclic if and only if $(g)_\#([Z, B]) \subset G^f(Z, X)$ for any space Z .*

Proof. Suppose that $g : B \rightarrow X$ is f -cyclic. Let Z be a space and $\theta : Z \rightarrow B$ a map. Since $g : B \rightarrow X$ is f -cyclic, there is a map $G : A \times B \rightarrow X$ such that $Gj \sim \nabla(f \vee g)$, where $j : A \vee B \rightarrow A \times B$ is the inclusion and $\nabla : X \vee X \rightarrow X$ is the folding map. Then $\phi = G(1 \times \theta) : A \times Z \rightarrow X$ satisfies $\phi j \sim \nabla(f \vee g\theta)$. Thus we have $g_\#([Z, B]) \subset G^f(Z, X)$ for any space Z . On the other hand, taking $Z = B$ and $1_B : B \rightarrow B \in [B, B]$. Since $g \sim g_\#(1_B) \in G^f(B, X)$, $g : B \rightarrow X$ is f -cyclic. $\square$

THEOREM 2.8. [5] *A space X is a C_k^f -space for a map $f : A \rightarrow X$ if and only if $G^f(Z, X) = [Z, X]$ for any space Z with $\text{cat } Z \leq k$.*

EXAMPLE 2.9.

- (1) The torus T^k is a space with $\text{mapcat}(T^k, X) \leq k$. It is well known fact that $\text{cat } T^k = k$. Thus we know, from Proposition 2.4, that $\text{mapcat}(T^k, X) \leq k$, that is, $(e_k^X)_\# : [T^k, P^k(\Omega X)] \rightarrow [T^k, X]$ is an epimorphism.
- (2) If a space Z satisfy $\text{cat } Z \leq k$, then Z is also a space with $\text{mapcat}(Z, X) \leq k$ from Proposition 2.4.

DEFINITION 2.10. *Let $f : A \rightarrow X$ be a map and Z a space with $\text{mapcat}(Z, X) \leq k$. Then a space X is called a WC_k^f -space with respect to a space Z if $(e_k^X)_\#([Z, P^k(\Omega X)]) \subset G^f(Z, X)$, where $e_k^X : P^k(\Omega X) \rightarrow X$ is the natural inclusion.*

THEOREM 2.11. *Let $f : A \rightarrow X$ be a map and Z space with $\text{mapcat}(Z, X) \leq k$. A space X is a WC_k^f -space with respect to Z if and only if $G^f(Z, X) = [Z, X]$.*

Proof. Suppose that X is a WC_k^f -space with respect to Z . Since $\text{mapcat}(Z, X) \leq k$, $(e_k^X)_\# : [Z, P^k(\Omega X)] \rightarrow [Z, X]$ is an epimorphism. Since X is a WC_k^f -space with respect to Z , $[Z, X] = (e_k^X)_\#([Z, P^k(\Omega X)]) \subset G^f(Z, X)$ and $G^f(Z, X) = [Z, X]$.

Conversely, assume that $G^f(Z, X) = [Z, X]$. Thus we know $(e_k^X)_\#([Z, P^k(\Omega X)]) = [Z, X] \subset G^f(Z, X)$ and X is a WC_k^f -space with respect to Z . $\square$

We have the following corollary from Theorem 2.8 and Example 2.9.

COROLLARY 2.12. *X is a C_k^f -space if and only if for each space Z with $\text{cat } Z \leq k$, X is a WC_k^f -space with respect to Z .*

THEOREM 2.13. *Let $f : A \rightarrow X$ and $g : B \rightarrow Y$ be any maps and Z a space with $\text{mapcat } (Z, X) \leq k$. Then the product space $X \times Y$ is a $WC_k^{f \times g}$ -space with respect to Z if and only if X is a WC_k^f -space with respect to Z and Y is a WC_k^g -space with respect to Z .*

Proof. Suppose $X \times Y$ is a $WC_k^{f \times g}$ -space with respect to Z . It is known [5] that $G^{f \times g}(Z, X \times Y) \cong G^f(Z, X) \times G^g(Z, Y)$ for any space Z . Since Z is a space with $\text{mapcat } (Z, X) \leq k$, we have, from Theorem 2.11, that $G^f(Z, X) \times G^g(Z, Y) \cong G^{f \times g}(Z, X \times Y) = [Z, X \times Y] = [Z, X] \times [Z, Y]$ and hence $G^f(Z, X) = [Z, X]$ and $G^g(Z, Y) = [Z, Y]$. Thus X is a WC_k^f -space with respect to Z and Y is a WC_k^g -space with respect to Z .

Conversely, suppose that X is a WC_k^f -space with respect to Z and Y is a WC_k^g -space with respect to Z . Then $G^f(Z, X) = [Z, X]$ and $G^g(Z, Y) = [Z, Y]$ by Theorem 2.11. It follows that $G^{f \times g}(Z, X \times Y) \cong G^f(Z, X) \times G^g(Z, Y) = [Z, X] \times [Z, Y] = [Z, X \times Y]$. Thus $X \times Y$ is a $WC_k^{f \times g}$ -space with respect to Z . $\square$

3. DWC_k^p -spaces with respect to spaces

In [3], Ganea introduced the concept of cocategory of a space as follows; Let X be a any space. Define a sequence of cofibrations

$$\mathcal{C}_k : X \xrightarrow{e'_k} F_k \xrightarrow{s'_k} B_k \quad (k \geq 0)$$

as follows, let $\mathcal{C}_0 : X \xrightarrow{e'_0} cX \xrightarrow{s'_0} \Sigma X$ be the standard cofibration. Assuming $\mathcal{C}_k$ to be defined, let F'_{k+1} be the fibre of s'_k and $e''_{k+1} : X \rightarrow F'_{k+1}$ lift e'_k . Define F_{k+1} as the reduced mapping cylinder of e''_{k+1} , let $e'_{k+1} : X \rightarrow F_{k+1}$ is the obvious inclusion map, and let $B_{k+1} = F_{k+1}/e'_{k+1}(X)$ and $s'_{k+1} : F_{k+1} \rightarrow B_{k+1}$ the quotient map.

DEFINITION 3.1. [3] *The cocategory of X , $\text{cocat } X$, is the least integer $k \geq 0$ for which there is a map $r : F_k \rightarrow X$ such that $r \circ e'_k \sim 1$. If there is no such integer, $\text{cocat } X = \infty$.*

The following remark can easily obtained from the above definition.

REMARK 3.2. *cocat* $X \leq k$ if and only if $e'_k : X \rightarrow F_k$ has a left homotopy inverse.

For a map $p : X \rightarrow A$, a based map $g : X \rightarrow B$ is *p-cocyclic* [9] if there is a map $\theta : X \rightarrow A \vee B$ such that $j\theta \sim (p \times g)\Delta$, where $j : A \vee B \rightarrow A \times B$ is the inclusion and $\Delta : X \rightarrow X \times X$ is the diagonal map. The dual Gottlieb set for a map $p : X \rightarrow A$, $DG^p(X, B)$, is the set of all homotopy classes of *p-cocyclic* maps from X to B . In the case $p = 1_X : X \rightarrow X$, we call a 1-cocyclic map is just a cocyclic map, and denoted by, $DG(X, B)$, which is the set of all homotopy classes of cocyclic maps from X to B .

In general, $DG(X, B) \subset DG^p(X, B) \subset [X, B]$ for any map $p : X \rightarrow A$ and any space B . However, there is an example in [13] such that $DG(X, B) \neq DG^p(X, B) \neq [X, B]$.

Let $g : X \rightarrow Z$ be a map. A cocategory of a map is less than equal to k , *cocat* $g \leq k$, [3] if there is a map $\bar{g} : F_k^X \rightarrow Z$ such that $\bar{g} \circ e'_k{}^X \sim g : X \rightarrow Z$.

DEFINITION 3.3. Let X, Z be any two spaces. The mapcocategory of mapping space from X to Z is less than equal to k , *mapcocat* $(X, Z) \leq k$, means that for any map $g : X \rightarrow Z$, *cocat* $g \leq k$.

It is clear that *mapcocat* $(X, Z) \leq k$ if and only if $(e'_k{}^X)^\# : [F_k^X, Z] \rightarrow [X, Z]$ is an epimorphism.

PROPOSITION 3.4. *cocat* $Z \leq k$ if and only if for any space X , *mapcocat* $(X, Z) \leq k$.

Proof. Suppose that *cocat* $Z \leq k$. Then there is a map $s'_k{}^Z : F_k^Z \rightarrow Z$ such that $s'_k{}^Z \circ e'_k{}^Z \sim 1$. Let X be a space and $g : X \rightarrow Z$ a map. We see $F_k(g) \circ e'_k{}^X \sim e'_k{}^Z \circ g$ by the naturality of the construction of F_k^Z as is shown in the following homotopy commutative diagram:

$$\begin{array}{ccc} F_k^X & \xrightarrow{F_k(g)} & F_k^Z \\ e'_k{}^X \uparrow & & e'_k{}^Z \uparrow \\ X & \xrightarrow{g} & Z. \end{array}$$

Thus we have a map $\bar{g} = s'_k{}^Z \circ F_k(g) : F_k^X \rightarrow Z$ such that $\bar{g} \circ e'_k{}^X \sim g$. Thus we know *mapcocat* $(X, Z) \leq k$. On the other hand, suppose that for any space X , the mapcocategory *mapcocat* $(X, Z) \leq k$. Taking $X = Z$ and $g = 1_Z$, then we know that *cocat* $Z \leq k$. $\square$

PROPOSITION 3.5. *cocat* $X \leq k$ if and only if for any space Z , *mapcocat* $(X, Z) \leq k$.

Proof. Suppose that $\text{cocat } X \leq k$. Then there is a map $s_k'^X : F_k^X \rightarrow X$ such that $s_k'^X \circ e_k'^X \sim 1$. Let Z be a space and $g : X \rightarrow Z$ a map. Then we have $(g \circ s_k'^X) \circ e_k'^X \sim 1_X \circ g \sim g$ and $\text{mapcat } (X, Z) \leq k$. On the other hand, suppose that for any space Z , the mapcocoategory $\text{mapcocat } (X, Z) \leq k$. Taking $Z = X$ and $g = 1_X$, then we know that $\text{cocat } X \leq k$. $\square$

It is introduced [18] that a space X is called DC_k^p -space for a map $p : X \rightarrow A$ if $e_k'^X : X \rightarrow F_k^X$ is p -cocyclic.

LEMMA 3.6. *Let $p : X \rightarrow A$ be a map. Then $g : X \rightarrow B$ is p -cocyclic if and only if $(g)^\#([B, Z]) \subset DG^p(X, Z)$ for any space Z .*

Proof. Suppose that $g : X \rightarrow B$ is p -cocyclic. Let Z be a space and $h : B \rightarrow Z$ a map. Since $g : X \rightarrow B$ is p -cocyclic, there is a map $\theta : X \rightarrow A \vee B$ such that $j\theta \sim (p \times g)\Delta$, where $j : A \vee B \rightarrow A \times B$ is the inclusion and $\Delta : X \rightarrow X \times X$ is the diagonal map. Then $\phi = (1 \vee h)\theta : X \rightarrow A \vee Z$ satisfies $j\phi \sim (p \times hg)\Delta$. Thus we have $g^\#([B, Z]) \subset DG^p(X, Z)$ for any space Z . On the other hand, taking $Z = B$ and $1_B : B \rightarrow B \in [B, B]$. Since $g \sim g^\#(1_B) \in DG^p(X, B)$, $g : X \rightarrow B$ is p -cocyclic. $\square$

THEOREM 3.7. [18] *A space X is a DC_k^p -space for a map $p : X \rightarrow A$ if and only if $DG^p(X, Z) = [X, Z]$ for any space Z with $\text{cocat } Z \leq k$.*

REMARK 3.8. *If a space Z satisfy $\text{cocat } Z \leq k$, then also Z is a space with $\text{mapcocat}(Z, X) \leq k$ from Proposition 3.4.*

DEFINITION 3.9. *Let $p : X \rightarrow A$ be a map and Z a space with $\text{mapcocat } (X, Z) \leq k$. A space X is called a DWC_k^p -space with respect to a space Z if $(e_k'^X)^\#([F_k^X, Z]) \subset DG^p(X, Z)$.*

THEOREM 3.10. *Let $p : X \rightarrow A$ be a map and Z a space with $\text{mapcocat } (X, Z) \leq k$. Then a space X is a DWC_k^p -space with respect to Z if and only if $DG^p(X, Z) = [X, Z]$.*

Proof. Suppose that X is a DWC_k^p -space with respect to Z . Since $\text{mapcocat } (X, Z) \leq k$, $(e_k'^X)^\# : [F_k^X, Z] \rightarrow [X, Z]$ is an epimorphism. Since X is a DWC_k^p -space with respect to Z , $[X, Z] = (e_k'^X)^\#([F_k^X, Z]) \subset DG^p(X, Z)$ and $DG^p(X, Z) = [X, Z]$.

Conversely, assume that $DG^p(X, Z) = [X, Z]$. Thus we know $(e_k'^X)^\#([F_k^X, Z]) = [X, Z] \subset DG^p(X, Z)$ and X is a DWC_k^p -space with respect to Z . $\square$

We have the following corollary from Theorem 3.7 and Remark 3.8.

COROLLARY 3.11. X is a DC_k^p -space if and only if for each space Z with $\text{cocat } Z \leq k$, X is a DWC_k^p -space with respect to Z .

Let $p : X \rightarrow A$ and $q : Y \rightarrow A$ be any maps. Then it is known [18] that the relation $DG^{\nabla(p \vee q)}(X \vee Y, B) \equiv DG^p(X, B) \times DG^q(Y, B)$ holds for any space B .

THEOREM 3.12. Let $p : X \rightarrow A$ and $q : Y \rightarrow A$ be any maps and Z a space with $\text{mapcocat } (X, Z) \leq k$. Then the wedge space $X \vee Y$ is a $DWC_k^{\nabla(p \vee q)}$ -space with respect to Z if and only if X is a DWC_k^p -space with respect to Z and Y is a DWC_k^q -space with respect to Z .

Proof. If $X \vee Y$ is a $DWC_k^{\nabla(p \vee q)}$ -space with respect to Z , then we know, from Theorem 3.10 and the above fact, that $DG^p(X, Z) \times DG^q(Y, Z) \equiv DG^{\nabla(p \vee q)}(X \vee Y, Z) = [X \vee Y, Z] \equiv [X, Z] \times [Y, Z]$. Then we have $DG^p(X, Z) = [X, Z]$ and $DG^q(Y, Z) = [Y, Z]$. Thus we know that X is a DWC_k^p -space with respect to Z and Y is a DWC_k^q -space with respect to Z . On the other hand, suppose that X is a DWC_k^p -space with respect to Z and Y is a DWC_k^q -space with respect to Z . Then $DG^p(X, Z) = [X, Z]$, $DG^q(Y, Z) = [Y, Z]$. Thus we know $DG^{\nabla(p \vee q)}(X \vee Y, Z) \equiv DG^p(X, Z) \times DG^q(Y, Z) = [X, Z] \times [Y, Z] \equiv [X \vee Y, Z]$. Thus $X \vee Y$ is a $DWC_k^{\nabla(p \vee q)}$ -space with respect to Z . $\square$

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