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ON PUTINAR'S MATRICIAL MODEL OPERATOR OF RANK 2

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Abstract. In this paper we study the Putinar's matricial model operator of rank 2 and provide some evidences for the validity of the conjecture in [8].

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *quasinormal* if $T^*T^2 = TT^*T$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if it has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. In general it is quite difficult to determine the subnormality of an operator by definition. An alternative description of subnormality is given by the Bram-Halmos criterion, which states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([3],[4, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1\ ). \tag{1.1}$$

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Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{1.2}$$

is positive.

We now review a few essential facts concerning weak subnormality that we will need to begin with. Note that the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \left(\begin{smallmatrix} T & A \\ 0 & B \end{smallmatrix}\right)$ is normal, i.e.,

$$\begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases} \tag{1.3}$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* if there exist operators $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}')$ such that the first two conditions in (1.3) hold:

$$[T^*, T] = AA^* \quad \text{and} \quad A^*T = BA^*, \tag{1.4}$$

or equivalently, there is an extension $\widehat{T}$ of T such that

$$\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f \quad \text{for all } f \in \mathcal{H}.$$

The operator $\widehat{T}$ is called a *partially normal extension* (briefly, p.n.e.) of T. We also say that $\widehat{T}$ in $\mathcal{L}(\mathcal{K})$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T. It is known ([6, Lemma 2.5 and Corollary 2.7]) that

$$\widehat{T} = \text{ m.p.n.e.}(T) \iff \mathcal{K} = \bigvee\{\widehat{T}^{*n}h : \ h \in \mathcal{H}, \ n = 0, 1\},$$

and the m.p.n.e.(T) is unique. For convenience, if $\widehat{T} =$ m.p.n.e.(T) is also weakly subnormal then we write $\widehat{T}^{(2)} := \widehat{\widehat{T}}$ and more generally, $\widehat{T}^{(n)} := \widehat{\widehat{T}^{(n-1)}}$. It was ([6], [5]) shown that

$$\text{2-hyponormal} \implies \text{weakly subnormal} \implies \text{hyponormal} \tag{1.5}$$

and the converses of both implications in (1.5) are not true in general. In particular, the following lemma is very useful in the sequel.

LEMMA 1.1. ([6], [5]) *Let* $T \in \mathcal{L}(\mathcal{H})$.

(a) *If* T *is weakly subnormal then the operator* A *in (1.4) can be taken as a positive operator;*

(b) *If* T *is weakly subnormal then* $\ker[T^*, T]$ *is invariant for* T*;*

(c) *For any* $k \geq 1$, T *is* $(k+1)$*-hyponormal if and only if* T *is weakly subnormal and* $\widehat{T} := m.p.n.e.(T)$ *is* k*-hyponormal.*

The self-commutator of an operator plays an important role in the study of subnormality. Subnormal operators with finite rank self-commutators have been extensively studied ([2], [9], [11], [16], [17], [18], [20], [21]). Particular attention has been paid to hyponormal operators with rank 1 or rank 2 self-commutators ([7], [10], [12], [13], [14], [16], [19], [22]). In particular, B. Morrel [10] showed that a pure subnormal operator with rank 1 self-commutator (pure means having no normal summand) is unitarily equivalent to a linear function of the unilateral shift.

It is worth to noticing that in view of (1.5) and Lemma 1 (a), Morrel's theorem gives that every weakly subnormal operator with rank 1 self-commutator is subnormal.

2. The main results

M. Putinar [15] gave a matricial model for the hyponormal operator $T \in \mathcal{L}(\mathcal{H})$ with finite rank self-commutator, in the cases where

$$\mathcal{H}_0 := \bigvee_{k=0}^{\infty} T^{*k}\big(\operatorname{ran}[T^*, T]\big) \text{ has finite dimension } d \quad \text{and} \quad \mathcal{H} = \bigvee_{n=0}^{\infty} T^n \mathcal{H}_0.$$

Let $G_n := \bigvee_{k=0}^{n} T^k \mathcal{H}_0 \quad (n \geq 0)$ and $\mathcal{H}_n := G_n \ominus G_{n-1} \quad (n \geq 1)$. If $\dim(\mathcal{H}_n) = \dim(\mathcal{H}_{n+1}) = d \ \ (n \geq 0)$, then T has the following two-diagonal structure relative to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots$ ([15]):

$$T = \begin{pmatrix} B_0 & 0 & 0 & 0 & \cdots \\ A_0 & B_1 & 0 & 0 & \cdots \\ 0 & A_1 & B_2 & 0 & \cdots \\ 0 & 0 & A_2 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.1}$$

where

$$\begin{cases} [T^*, T] = ([B_0^*, B_0] + A_0^* A_0) \oplus 0_\infty; \\ [B_{n+1}^*, B_{n+1}] + A_{n+1}^* A_{n+1} = A_n A_n^* \quad (n \geq 0); \\ A_n^* B_{n+1} = B_n A_n^* \quad (n \geq 0). \end{cases} \tag{2.2}$$

We will refer the operator (2.1) to the *Putinar's matricial model operator of rank d*. This model was also introduced in [7], [12], [19], [20], and etc.

In [8], using the Agler's characterization of subnormality [1], the authors showed the following theorems:

THEOREM 2.1. *([8]) Let $T \in \mathcal{L}(\mathcal{H})$. If*

(i) *T is a pure hyponormal operator;*
(ii) *$[T^*, T]$ is of rank 2; and*
(iii) *$\ker [T^*, T]$ is invariant for T,*

then the following hold:

1. *If $T|_{\ker [T^*,T]}$ has the rank 1 self-commutator then T is subnormal;*
2. *If $T|_{\ker [T^*,T]}$ has the rank 2 self-commutator then T is either a subnormal operator or the Putinar's matricial model operator of rank 2.*

THEOREM 2.2. *([8]) The operator T in (2.1) is subnormal if B_n is normal for some $n \geq 0$.*

Also, they conjectured that:

CONJECTURE 2.3. *([8]) The Putinar's matricial model operator of rank 2 is subnormal.*

In this paper we examine the validity of the Conjecture 2.3, and we provide some affirmative evidences for the Conjecture 2.3. If A_0 and A_1 in (2.1) commute, we then have :

THEOREM 2.4. *Let T be the Putinar's matricial model operator of rank 2. If A_0 and A_1 in (2.1) commute then T is either subnormal or is of the following form by a translation or a multiplication by an appropriate scalar: $A_j = \left(\begin{smallmatrix} p_j & 0 \\ 0 & q_j \end{smallmatrix}\right)$ and $B_j = \left(\begin{smallmatrix} 0 & b_j \\ c_j & 0 \end{smallmatrix}\right)$ for $j = 0, 1, \cdots$, that is,*

$$T = \begin{pmatrix} 0 & b_0 & 0 & 0 & \dots \\ c_0 & 0 & 0 & 0 & \dots \\ p_0 & 0 & 0 & b_1 & \dots \\ 0 & q_0 & c_1 & 0 & \dots \\ 0 & 0 & p_1 & 0 & \dots \\ 0 & 0 & 0 & q_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.3}$$

Proof. Let

$$T_n := \begin{pmatrix} B_n & 0 & 0 & 0 & \dots \\ A_n & B_{n+1} & 0 & 0 & \dots \\ 0 & A_{n+1} & B_{n+2} & 0 & \dots \\ 0 & 0 & A_{n+2} & B_{n+3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (n = 0, 1, \cdots).$$

By [8], we can see that T_n is the minimal partially normal extension of T_{n+1} for each $n \geq 0$. Thus, by Lemma 1.1 (a), we can assume that A_n is positive for each $n \geq 0$.
Since A_0 and A_1 are diagonalizable and A_0 and A_1 commute, we can see that A_0 and A_1 are simultaneously diagonalizable. So we can write

$$A_n := \begin{pmatrix} p_n & 0 \\ 0 & q_n \end{pmatrix} \quad (n = 0, 1).$$

Also write

$$B_n := \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad (n = 0, 1).$$

By the third equality of (2.2), we have

$$\begin{cases} a_0 = a_1 =: a; \\ d_0 = d_1 =: d; \\ p_0 b_1 = b_0 q_0; \\ c_0 p_0 = q_0 c_1 . \end{cases}$$

If $a = d$ then by a translation we have

$$B_n = \begin{pmatrix} 0 & b_n \\ c_n & 0 \end{pmatrix} \quad (n = 0, 1).$$

So by the third equality of (2.2), B_2 is skew diagonal and in turn, by the second equality of (2.2), A_2 is diagonal. Repeating this argument with a telescoping method shows that B_n is skew diagonal and A_n is diagonal for each $n = 0, 1, \cdots$. Thus T is of the form (2.3).

Now suppose $a \neq d$. By a translation and a multiplication by an appropriate scalar, write

$$B_n := \begin{pmatrix} a & b_n \\ c_n & 0 \end{pmatrix} \quad (a \in \mathbb{R}, a \neq 0, n = 0, 1).$$

By the second equality of (2.2),

$$[B_1^*, B_1] = \begin{pmatrix} |c_1|^2 - |b_1|^2 & ab_1 - a\overline{c_1} \\ a\overline{b_1} - ac_1 & |b_1|^2 - |c_1|^2 \end{pmatrix}$$

is diagonal, and hence $b_1 = \overline{c_1}$. Thus B_1 is normal. Therefore, by Theorem 2.2, T is subnormal. $\square$

We now give general sufficient conditions for the subnormality of T in (2.3).

THEOREM 2.5. *The operator T in (2.3) is subnormal if one of the following holds:*

(i) $p_n \geq q_n$ *(or $q_n \geq p_n$) for all $n = m, m+1, \cdots$;*
(ii) $q_n \geq |c_n|$ *and* $p_n \geq |b_n|$ *for some $n \geq 0$;*
(iii) $|b_n| = |c_n|$ *for some $n \geq 0$;*
(iv) $p_n = q_n$ *for some $n \geq 0$;*
(v) $p_n = p_{n+1}$ *(or $q_n = q_{n+1}$) for some $n \geq 0$;*
(vi) $|b_n| = |b_{n+1}|$ *(or $|c_n| = |c_{n+1}|$) for some $n \geq 0$.*

Proof. First of all, observe that from the second and third equalities of (2.2),

$$\begin{cases} p_{n+1}^2 = p_n^2 + |b_{n+1}|^2 - |c_{n+1}|^2; \\ q_{n+1}^2 = q_n^2 - |b_{n+1}|^2 + |c_{n+1}|^2; \\ p_n b_{n+1} = b_n q_n; \\ c_n p_n = q_n c_{n+1}. \end{cases} \tag{2.4}$$

(i) Without loss of generality we may assume $p_n \geq q_n$ for all $n = 0, 1, \cdots$. Thus $\{|b_n|\}$ is decreasing and $\{|c_n|\}$ is increasing. By using the fourth recursive formula of (2.4) repeatedly, we have

$$c_{n+1} = \left(\prod_{j=0}^{n} \frac{p_j}{q_j} \right) c_0.$$

Since $\frac{p_j}{q_j} \geq 1$ for each $j \geq 0$, the sequence $\{|c_n|\}$ should converge, so that $\sum_{j=0}^{\infty} \mathrm{Log}\,(\frac{p_j}{q_j})$ converges, and hence the sequence $\{\frac{p_j}{q_j}\}$ converges to 1. Similarly, the sequence $\{|b_n|\}$ converges. Say $b := \lim |b_n|$ and $c := \lim |c_n|$. We now claim that $b = c$. Assume to the contrary that $c > b$ and let $\epsilon := c^2 - b^2 > 0$. Then there exists $N \in \mathbb{Z}_+$ such that $|c_{n+1}|^2 - |b_{n+1}|^2 \geq \frac{\epsilon}{2}$ for all $n \geq N$. Then by the second equality of (2.4), if $n \geq N$ then

$$q_{n+m}^2 \geq q_n^2 + \frac{\epsilon}{2} m \to \infty \text{ as } m \to \infty,$$

which implies that the sequence $\{q_n\}$ is unbounded, a contradiction. If instead $b > c$ then again the sequence $\{p_n\}$ is unbounded, a contradiction. This proves $b = c$. Since $\{|b_n|\}$ is decreasing and $\{|c_n|\}$ is increasing, we can see that

$$b_n \geq c_n \quad \text{for all } n \geq 0. \tag{2.5}$$

Thus by (2.4) and (2.5) we can conclude that $\{p_n\}$ is increasing and $\{q_n\}$ is decreasing. So both $\{p_n\}$ and $\{q_n\}$ converge. But since $\frac{p_n}{q_n} \to 1$, we can say $p_n \to p$ and $q_n \to p$ for some $p > 0$. So

$$p_0 \leq p_1 \leq p_2 \leq \cdots \leq p \leq \cdots \leq q_2 \leq q_1 \leq q_0.$$

But since $p_0 \geq q_0$ it follows that $p_n = p = q_n$ for all $n \geq 0$. By (2.5), this also implies that $|b_n| = |c_n|$ for all $n \geq 0$. Therefore all the B_n are normal. By Theorem 2.2, we can conclude that T is subnormal.

(ii) Without loss of generality, we may assume that $q_0 \geq |c_0|$ and $p_0 \geq |b_0|$. If we put

$$A_{-1} := \left([B_0^*, B_0] + A_0^2\right)^{\frac{1}{2}} \qquad \text{and} \quad B_{-1} := A_{-1} B_0 A_{-1}^{-1}$$

then $\widehat{T} := \begin{pmatrix} B_{-1} & 0 \\ A_{-1} & T \end{pmatrix} =$ m.p.n.e.(T). So we need to show that

$$[\widehat{T}^*, \widehat{T}] = \left([B_{-1}^*, B_{-1}] + A_{-1}^2\right) \oplus 0_\infty \geq 0.$$

A straightforward calculation shows that

$$A_{-1} = \begin{pmatrix} |c_0|^2 - |b_0|^2 + p_0^2 & 0 \\ 0 & |b_0|^2 - |c_0|^2 + q_0^2 \end{pmatrix} \qquad \text{and} \quad B_{-1} = \begin{pmatrix} 0 & \frac{p_{-1}}{q_{-1}} b_0 \\ \frac{q_{-1}}{p_{-1}} c_0 & 0 \end{pmatrix},$$

where

$$p_{-1} := \left(|c_0|^2 - |b_0|^2 + p_0^2\right)^{\frac{1}{2}} \qquad \text{and} \quad q_{-1} := \left(|b_0|^2 - |c_0|^2 + q_0^2\right)^{\frac{1}{2}}.$$

So

$$\begin{aligned} &[B_{-1}^*, B_{-1}] + A_{-1}^2 \\ &= \begin{pmatrix} \left(\frac{q_{-1}}{p_{-1}}\right)^2 |c_0|^2 - \left(\frac{p_{-1}}{q_{-1}}\right)^2 |b_0|^2 + p_{-1}^2 & 0 \\ 0 & \left(\frac{p_{-1}}{q_{-1}}\right)^2 |b_0|^2 - \left(\frac{q_{-1}}{p_{-1}}\right)^2 |c_0|^2 + q_{-1}^2 \end{pmatrix}. \end{aligned}$$

Observe that if $q_0 \geq |c_0|$ then

$$\begin{aligned} &\left(\frac{q_{-1}}{p_{-1}}\right)^2 |c_0|^2 - \left(\frac{p_{-1}}{q_{-1}}\right)^2 |b_0|^2 + p_{-1}^2 \\ &= \frac{1}{p_{-1}^2 q_{-1}^2} \left(q_{-1}^4 |c_0|^2 - p_{-1}^4 |b_0|^2 + q_{-1}^2 p_{-1}^4\right) \\ &= \frac{1}{p_{-1}^2 q_{-1}^2} \left(q_{-1}^4 |c_0|^2 + p_{-1}^4 (q_0^2 - |c_0|^2)\right) \\ &\geq 0 \end{aligned}$$

and similarly, if $p_0 \geq |b_0|$ then

$$\left(\frac{p_{-1}}{q_{-1}}\right)^2 |b_0|^2 - \left(\frac{q_{-1}}{p_{-1}}\right)^2 |c_0|^2 + q_{-1}^2 \geq 0,$$

and therefore $[B_{-1}^*, B_{-1}] + A_{-1}^2 \geq 0$. So T is 2-hyponormal. But since if we put

$$b_{-1} := \frac{p_{-1}}{q_{-1}} b_0 \quad \text{and} \quad c_{-1} := \frac{q_{-1}}{p_{-1}} c_0$$

then

$$p_{-1}^2 - |b_{-1}|^2 = \frac{p_{-1}^2}{q_{-1}^2}\left(q_{-1}^2 - |b_0|^2\right) = \frac{p_{-1}^2}{q_{-1}^2}\left(q_0^2 - |c_0|^2\right) \geq 0$$

and

$$q_{-1}^2 - |c_{-1}|^2 = \frac{q_{-1}^2}{p_{-1}^2}\left(p_{-1}^2 - |c_0|^2\right) = \frac{q_{-1}^2}{p_{-1}^2}\left(p_0^2 - |b_0|^2\right) \geq 0,$$

we can repeat the above argument. Thus T is k-hyponormal for every $k \in \mathbb{Z}_+$ and hence T is subnormal.

(iii) Since $|b_n| = |c_n|$ for some $n \geq 0$, B_n is normal. Thus, it follows from Theorem 2.2.

(iv) Without loss of generality, we may assume $p_0 = q_0$. So we can write $A_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and hence $B_0 = B_1$ by the third equality of (2.2). Now if we define

$$A_{-1} := \left([B_0^*, B_0] + A_0^2\right)^{\frac{1}{2}} \quad \text{and} \quad B_{-1} := A_{-1} B_0 A_{-1}^{-1},$$

then $\widehat{T} := \left(\begin{smallmatrix} B_{-1} & 0 \\ A_{-1} & T \end{smallmatrix}\right) =$ m.p.n.e.(T). So we need to shows that

$$[\widehat{T}^*, \widehat{T}] = \left([B_{-1}^*, B_{-1}] + A_{-1}^2\right) \oplus 0_\infty \geq 0.$$

A straightforward calculation shows that

$$A_{-1} = P^{-1} A_1 P \quad \text{and} \quad B_{-1} = B_2$$

where $P := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. So

$$[B_{-1}^*, B_{-1}] + A_{-1}^2 = P^{-1} A_2^2 P \geq 0.$$

Thus we see that T is 2-hyponormal. Similarly, we can repeat this backward extension. Therefore we can conclude that T is subnormal.

(v) If $p_n = p_{n+1}$ (or $q_n = q_{n+1}$), then by the first (or second) equality of (2.4) we have $|b_{n+1}| = |c_{n+1}|$. Therefore the result follows from (iv).

(vi) If $|b_n| = |b_{n+1}| \neq 0$ (or $|c_n| = |c_{n+1}| \neq 0$), then by the third (or fourth) equality of (2.4) we have $p_n = q_n$. Therefore the result follows from (iii). If instead $|b_n| = |b_{n+1}| = 0$ (or $|c_n| = |c_{n+1}| = 0$), then by the second (or first) equality of (2.4), we have $q_{n+1} \geq |c_{n+1}|$ (or $p_{n+1} \geq |b_{n+1}|$). Therefore the result follows from (ii). $\square$

We thus have :

Theorem 2.6. *Let T be the Putinar's matricial model operator of rank 2. If the matrix B_j in (2.1) is of the form $B_j = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$ for some $a \in \mathbb{C}$ and for some $j \geq 0$ then T is subnormal.*

Proof. Without loss of generality we can assume $j = 0$. By Lemma 1.1 (a), we can also write $A_0 = \begin{pmatrix} p_0 & 0 \\ 0 & q_0 \end{pmatrix}$. From the third equality of (2.2) we can see that B_1 is of the form $B_1 = \left(\begin{smallmatrix} 0 & b_1 \\ 0 & 0 \end{smallmatrix}\right)$ for some $b_1 \in \mathbb{C}$. Let $A_1 \equiv \left(\begin{smallmatrix} p_1 & r_1 \\ \overline{r_1} & q_1 \end{smallmatrix}\right)$be positive. Then by the second equality of (2.2), we have $r_1 = 0$. It thus follows that A_1 is a diagonal matrix. Therefore T is subnormal by Theorem 2.4 and Theorem 2.5 (vi). $\square$

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